Hamiltonian and Brownian systems with long-range interactions: IV. General kinetic equations from the quasilinear theory

Pierre-Henri Chavanis

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Laboratoire de Physique Théorique (CNRS UMR 5152),
Université Paul Sabatier,
118, route de Narbonne, 31062 Toulouse Cedex 4, France
E-mail: *chavanis@irsamc.ups-tlse.fr*

Abstract

We develop the kinetic theory of Hamiltonian systems with weak long-range interactions. Starting from the Klimontovich equation and using a quasilinear theory, we obtain a general kinetic equation that can be applied to spatially inhomogeneous systems and that takes into account memory effects. This equation is valid at order $1/N$ in a proper thermodynamic limit and it coincides with the kinetic equation obtained from the BBGKY hierarchy. For $N \rightarrow +\infty$, it reduces to the Vlasov equation describing collisionless systems. We describe the process of phase mixing and violent relaxation leading to the formation of a quasi stationary state (QSS) on the coarse-grained scale. We interprete the physical nature of the QSS in relation to Lynden-Bell's statistical theory and discuss the problem of incomplete relaxation. In the second part of the paper, we consider the relaxation of a test particle in a thermal bath. We derive a Fokker-Planck equation by directly calculating the diffusion tensor and the friction force from the Klimontovich equation. We give general expressions of these quantities that are valid for possibly spatially inhomogeneous systems with long correlation time. We show that the diffusion and friction terms have a very similar structure given by a sort of generalized Kubo formula. We also obtain non-markovian kinetic equations that can be relevant when the auto-correlation function of the force decreases slowly with time. An interest of our approach is to develop a formalism that remains in physical space (instead of Fourier space) and that can deal with spatially inhomogeneous systems.

1 Introduction

In the treatment of physical systems in nature, a fundamental distinction must be made between systems for which the interaction between particles is short-range or long-range. Systems with short-range interactions have been studied for a very long time. They are spatially homogeneous, the ordinary thermodynamic limit ($N \to +\infty$ with fixed N/V) applies, and the statistical ensembles are equivalent for $N \to +\infty$. In recent years, a growing number of physical systems with truly long-range interactions have emerged and have been actively studied by different groups. An impetus has been given by the conference in Les Houches in 2002 [1] which showed the connections and the analogies between different topics: astrophysics, two-dimensional hydrodynamics, plasma physics,... Systems with long-range interactions have very peculiar properties. They can be spatially inhomogeneous (due to the spontaneous formation of coherent structures), the ordinary thermodynamic limit $N \to +\infty$ with fixed N/V does not apply and the statistical ensembles are generically inequivalent. This does not mean that statistical mechanics breaks down for these systems, but simply that it must be reformulated so as to take into account their peculiarities. Therefore, we must go back to the foundations and to the basic principles of statistical mechanics, thermodynamics and kinetic theory.

In previous papers of this series [2, 3, 4], we have developed a statistical mechanics and a kinetic theory adapted to systems with weak long-range interactions. Our approach heavily relies on many important works that have been developed in astrophysics for stellar systems [5, 6, 7], in hydrodynamics for two-dimensional vortices [8, 9, 10] and in plasma physics [11, 12]. However, the originality of our approach is to remain as general as possible and develop a formalism that can be applied to a wide variety of systems with long-range interactions. This includes important toy models like the HMF model [13], for example, where explicit analytical results can be obtained. However, our approach is more general and aims at showing the unity of the subject and the connection between different systems, emphasizing their analogies and differences. In this sense, we extend our original approach [14, 10] where we first showed the analogy between the statistical mechanics of stellar systems and two-dimensional vortices, which are two physical systems of considerable interest.

In this paper, we consider material particles having inertia [1] and interacting via a *weak* long-range binary potential of interaction $u(|\mathbf{r} - \mathbf{r}'|)$ in a space of dimension d. In Paper I, we have determined the statistical equilibrium states and the static correlation functions in a properly defined thermodynamic limit. For attractive potentials, we have shown the existence of a critical energy E_c (in the microcanonical ensemble) or a critical temperature T_c (in the canonical ensemble) separating a spatially homogeneous phase from a spatially inhomogeneous phase. In Paper II, using an analogy with plasma physics, we have developed a kinetic theory of systems with long-range interactions in the spatially homogeneous phase. In Paper III, we have studied the growth of correlations from the BBGKY hierachy and the connection to the kinetic theory. In the present paper, we further develop the kinetic theory of Hamiltonian systems with long-range interactions by starting from the Klimontovitch equation and using a quasilinear theory. We derive general kinetic equations that can be applied to spatially inhomogeneous systems and that take into account memory effects. These peculiarities are specific to systems with unshielded long-range interactions and are *novel* with respect to the much more studied case of spatially homogeneous systems with short-range (or shielded) interactions. However, we show that when the system is spatially homogeneous and when memory effects can be neglected, we recover the familiar kinetic equations of plasma physics [2] discussed in Paper II.

[1] The case of 2D point vortices, that have no inertia, is special and must be treated separately [15].

[2] The kinetic equations discussed in [3] have the same form as the Landau and Lenard-Balescu equations of plasma physics except that the Fourier transform of the potential of interaction $\hat{u}_{plasma}(k) \sim e^2/k^2$ is replaced by

This paper is organized as follows. In Sec. 2.1, we derive a general kinetic equation from the Klimontovich equation by using a quasilinear theory. This equation is valid at order $O(1/N)$ in the proper thermodynamic limit $N \to +\infty$ defined in Paper I. It coincides with the kinetic equation obtained from the BBGKY hierarchy in Paper III. For $N \to +\infty$, this kinetic equation reduces to the Vlasov equation. At order $O(1/N)$ it takes into account the effect of "collisions" (more properly "correlations") between particles due to finite N effects. It describes therefore the evolution of the system on a timescale Nt_D, where t_D is the dynamical time. This general kinetic equation applies to systems that can be spatially inhomogeneous and takes into account non-markovian effects. However, in order to obtain a closed kinetic equation, we have been obliged to neglect some collective effects. This is the main drawback of our approach: a more general treatment should take into account both spatial inhomogeneity and collective effects. If we restrict ourselves to spatially homogeneous systems and neglect memory terms, we recover the Landau equation as a special case. Therefore, the collective effects that we have neglected correspond to the effects of polarization taken into account in the Lenard-Balescu equation when the system is homogeneous (in plasma physics, they lead to Debye shielding). In Sec. 2.2, we develop a quasilinear theory of the Vlasov equation [17, 18, 19] in relation with the process of violent relaxation [20, 21] in the collisionless regime of the dynamics. We derive a kinetic equation for the coarse-grained distribution function $\overline{f}(\mathbf{r}, \mathbf{v}, t)$ and use this equation to describe the problem of *incomplete relaxation* [22] leading to deviations from the Lynden-Bell distribution. We show the analogies and the differences between the quasilinear theory of the Vlasov equation used to describe the process of violent collisionless relaxation and the quasilinear theory of the Klimontovich equation used to describe the process of slow collisional relaxation. In Sec. 3, we consider the relaxation of a test particle in a bath of field particles. The relaxation of the test particle is due to the combined effect of a diffusion term and a friction term. We derive the diffusion coefficient from the Kubo formula and the friction term from a linear response theory based on the Klimontovich equation. Like in the previous sections, the originality of our approach is to develop a formalism that can describe spatially inhomogeneous systems and that can take into account memory terms. If we consider spatially homogeneous systems with short memory, we recover the results obtained in Paper II. However, spatial inhomogeneity and memory effects can be important in systems with long-range interactions. Therefore, in Sec. 4 we derive non-markovian kinetic equations that generalize the standard Fokker-Planck equations. We consider explicit applications to self-gravitating systems and to the HMF model.

2 Kinetic equations from a quasilinear theory

In this section, we obtain a general kinetic equation (13) describing the collisional evolution of a Hamiltonian system of particles with weak long-range interactions. This equation, derived from a quasilinear theory of the Klimontovich equation, is valid at order $O(1/N)$ in the proper thermodynamics limit $N \to +\infty$ defined in Paper I. Then, we discuss the analogies and the differences with the quasilinear theory of the Vlasov equation developed in [17, 18, 19] to

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a more general potential $\hat{u}(k)$ that can take negative values in the case of *attractive* interactions. This change of sign is crucial for the stability of the homogeneous phase and is responsible for instabilities (similar to the Jeans instability) for $T < T_c$ or $E < E_c$. In the case where the homogeneous phase is stable (for $T > T_c$ or $E > E_c$), the potential $\hat{u}(k)$ enters explicitly in the Lenard-Balescu equation (II-49) through the dielectric function and the results can be different from those obtained with the Coulombian potential $\hat{u}_{plasma}(k)$. However, when collective effects are ignored, we get the Landau equation (II-40) where the potential of interaction $\hat{u}(k)$ appears only in a multiplicative constant (II-43) controlling the timescale of the relaxation. Note finally that the results of the kinetic theory depend on the dimension of space d [16].

describe the process of violent relaxation [20, 21] in the collisionless regime.

2.1 The slow collisional relaxation

The exact distribution function (DF) of a system of particles in interaction is a sum of Dirac functions

$$f_d(\mathbf{r}, \mathbf{v}, t) = \sum_i m\delta(\mathbf{r} - \mathbf{r}_i(t))\delta(\mathbf{v} - \mathbf{v}_i(t)), \tag{1}$$

satisfying the Klimontovich equation

$$\frac{\partial f_d}{\partial t} + \mathbf{v}\frac{\partial f_d}{\partial \mathbf{r}} - \nabla\Phi_d \frac{\partial f_d}{\partial \mathbf{v}} = 0, \tag{2}$$

where $\Phi_d(\mathbf{r}, t) = \int u(|\mathbf{r} - \mathbf{r}'|) f_d(\mathbf{r}', \mathbf{v}', t) d\mathbf{r}' d\mathbf{v}'$ is the exact potential created by f_d. The Klimontovich equation (2) should not be confused with the Vlasov equation (14) which has the same mathematical structure but which applies to the *smooth* distribution function f. The Vlasov equation is valid during the collisionless regime (see Sec. 2.2) while the Klimontovich equation is exact and contains the same information as the Hamiltonian equations (I-1). We now decompose the exact distribution function in the form $f_d = f + \delta f$ where $f = \langle f_d \rangle$ is the smooth distribution function and δf the fluctuation around it. Substituting this decomposition in Eq. (2) and locally averaging over the fluctuations, we get

$$\frac{\partial f}{\partial t} + Lf = \left\langle \nabla\delta\Phi \frac{\partial \delta f}{\partial \mathbf{v}} \right\rangle, \tag{3}$$

where $L = \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \nabla\Phi \frac{\partial}{\partial \mathbf{v}}$ is an advection operator in phase space constructed with the smooth field. Subtracting Eq. (3) from Eq. (2) and neglecting non linear terms in the fluctuations [3], we obtain the following equation for the evolution of the fluctuations

$$\frac{\partial \delta f}{\partial t} + L\delta f = \nabla\delta\Phi \frac{\partial f}{\partial \mathbf{v}}. \tag{4}$$

Equations (3) and (4) form the basis of the quasilinear theory. For spatially homogeneous systems, they can be solved with the aid of Laplace-Fourier transforms and they yield the Lenard-Balescu equation (see, e.g., [12] and Appendix B of Paper II). In the present work, we shall proceed differently so as to treat the case of systems that are not necessarily spatially homogeneous and not necessarily markovian. Our method avoids the use of Laplace-Fourier transforms and remains in physical space. This yields expressions with a clear interpretation which enlightens the basic physics. The drawback of our approach, however, is that it neglects collective effects. The formal solution of Eq. (4) is

$$\delta f(t) = G(t, 0)\delta f(0) + \int_0^t d\tau G(t, t-\tau)\nabla\delta\Phi(t-\tau)\frac{\partial f}{\partial \mathbf{v}}(t-\tau), \tag{5}$$

[3] As shown in Papers I and III, the proper thermodynamic limit corresponds to $N \to +\infty$ in such a way that the coupling constant $u_* \sim 1/N$ while the individual mass $m \sim 1$, the temperature $\beta \sim 1$, the energy per particle $E/N \sim 1$ and the volume $V \sim 1$ are fixed. This implies that $|\mathbf{r}| \sim 1$, $|\mathbf{v}| \sim 1$. We also have $f/N \sim 1$ and $\delta f/N \sim 1/\sqrt{N}$ so that $\Phi \sim u_* f \sim 1$ and $\delta\Phi \sim u_*\delta f \sim 1/\sqrt{N}$. With these scalings, we see that the terms that we have kept in Eq. (4) are of order $\delta f \sim \sqrt{N}$ and $f\delta\Phi \sim \sqrt{N}$ while the nonlinear terms that we have neglected are of order $\delta f \delta\Phi \sim 1 \ll \sqrt{N}$. We also note that the l.h.s. of Eq. (3) is of order $f \sim N$ while the r.h.s. of Eq. (3) is of order $\delta f\delta\Phi \sim 1$. It would have been more relevant to work in terms of the normalized distribution function $F = f/N$. Then Eq. (3) can be rewritten $\partial_t F + LF = (1/N)C(F)$ where the advective term is of order $O(1)$ and the collision term is of order $1/N$. Therefore, this equation describes the evolution of the system on a timescale $\sim Nt_D$. For $N \to +\infty$, it reduces to the Vlasov equation $\partial_t F + LF = 0$.

where G is the Green function associated with the advection operator L and we have noted $f(t) = f(\mathbf{r}, \mathbf{v}, t)$ and $\delta\Phi(t) = \delta\Phi(\mathbf{r}, t)$ for brevity. On the other hand, the perturbation of the potential is related to the perturbation of the distribution function through

$$-\nabla\delta\Phi(t) = \frac{1}{m}\int \mathbf{F}(1 \to 0)\delta f_1(t) d\mathbf{x}_1, \tag{6}$$

where 0 refers to the position $\mathbf{r}$ and we have noted $\delta f_1(t) = \delta f(\mathbf{r}_1, \mathbf{v}_1, t)$. Therefore, considering Eqs. (5) and (6), we see that the fluctuation of the field $\nabla\delta\Phi(t)$ is given by an iterative process: $\nabla\delta\Phi(t)$ depends on $\delta f_1(t)$ which itself depends on $\nabla\delta\Phi_1(t-\tau)$ etc. We shall solve this problem perturbatively in the thermodynamic limit $N \to +\infty$. To leading order, we get

$$\begin{aligned}
\left\langle \nabla\delta\Phi \frac{\partial \delta f}{\partial \mathbf{v}} \right\rangle &= -\frac{1}{m}\frac{\partial}{\partial v^\mu}\int d\mathbf{x}_1 F^\mu(1 \to 0) G_1(t,0) G(t,0) \langle \delta f_1(0) \delta f(0) \rangle \\
&+ \frac{1}{m^2}\frac{\partial}{\partial v^\mu}\int_0^t d\tau \int d\mathbf{x}_1 d\mathbf{x}_2 F^\mu(1 \to 0) G_1(t, t-\tau) G(t, t-\tau) \\
&\times \left\{ F^\nu(2 \to 0) \langle \delta f_1(t-\tau) \delta f_2(t-\tau) \rangle \frac{\partial f}{\partial v^\nu}(t-\tau) \right. \\
&\left. + F^\nu(2 \to 1) \langle \delta f(t-\tau) \delta f_2(t-\tau) \rangle \frac{\partial f_1}{\partial v_1^\nu}(t-\tau) \right\}.
\end{aligned} \tag{7}$$

Now, the fluctuation is exactly defined by

$$\delta f(\mathbf{r}, \mathbf{v}, t) = \sum_i m\delta(\mathbf{r} - \mathbf{r}_i(t))\delta(\mathbf{v} - \mathbf{v}_i(t)) - f(\mathbf{r}, \mathbf{v}, t). \tag{8}$$

Therefore, we obtain

$$\begin{aligned}
\langle \delta f_1 \delta f_2 \rangle = \langle \sum_{i \neq j} m^2 \delta(\mathbf{x}_1 - \mathbf{x}_i)\delta(\mathbf{x}_2 - \mathbf{x}_j) \rangle + \langle \sum_i m^2 \delta(\mathbf{x}_1 - \mathbf{x}_i)\delta(\mathbf{x}_2 - \mathbf{x}_i) \rangle \\
- \langle \sum_i m\delta(\mathbf{x}_1 - \mathbf{x}_i) f_2 \rangle - \langle \sum_j m\delta(\mathbf{x}_2 - \mathbf{x}_j) f_1 \rangle + f_1 f_2.
\end{aligned} \tag{9}$$

To evaluate the correlation function, we average with respect to the smooth distribution $f_i/(Nm)$ or $f_i f_j/(Nm)^2$. This operation leads to

$$\langle \delta f_1 \delta f_2 \rangle = \frac{N-1}{N} f_1 f_2 + m f_1 \delta(\mathbf{x}_1 - \mathbf{x}_2) - f_1 f_2 - f_2 f_1 + f_1 f_2, \tag{10}$$

so that, finally,

$$\langle \delta f_1 \delta f_2 \rangle = m f_1 \delta(\mathbf{x}_1 - \mathbf{x}_2) - \frac{1}{N} f_1 f_2. \tag{11}$$

Substituting this result in Eq. (7), we find that

$$\begin{aligned}
\left\langle \nabla\delta\Phi \frac{\partial \delta f}{\partial \mathbf{v}} \right\rangle &= \langle F^\mu(1 \to 0) \rangle \frac{\partial f}{\partial v^\mu} + \frac{1}{m}\frac{\partial}{\partial v^\mu}\int_0^t d\tau \int d\mathbf{x}_1 F^\mu(1 \to 0) G(t, t-\tau) \\
&\times \left\{ \mathcal{F}^\nu(1 \to 0) f_1(t-\tau) \frac{\partial f}{\partial v^\nu}(t-\tau) + \mathcal{F}^\nu(0 \to 1) f(t-\tau) \frac{\partial f_1}{\partial v_1^\nu}(t-\tau) \right\},
\end{aligned} \tag{12}$$

where we have regrouped the two Greenians G and G_1 in a single notation for brevity. Finally, replacing this expression in Eq. (3), we obtain the kinetic equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{N-1}{N}\langle \mathbf{F}\rangle\frac{\partial f}{\partial \mathbf{v}}=m\frac{\partial}{\partial v^{\mu}}\int_0^t d\tau\int d\mathbf{r}_1 d\mathbf{v}_1\frac{F^{\mu}}{m}(1\to 0)G(t,t-\tau)$$
$$\times\left\{\frac{\mathcal{F}^{\nu}}{m}(1\to 0)f_1\frac{\partial f}{\partial v^{\nu}}+\frac{\mathcal{F}^{\nu}}{m}(0\to 1)f\frac{\partial f_1}{\partial v_1^{\nu}}\right\}_{t-\tau}. \tag{13}$$

This is identical to the general kinetic equation (33) of Paper III obtained from the BBGKY hierarchy (or from the projection operator formalism [23]). We note that the term of order $1/N$ in the l.h.s. comes from the first term in Eq. (5). It corresponds to the mere advection of the fluctuations by the smooth field in Eq. (4), i.e. ignoring the coupling between the fluctuations of the field and the smooth distribution function (r.h.s. of Eq. (4)) which gives rise to the collision term.

2.2 The violent collisionless relaxation

To leading order in $N\to+\infty$, the smooth distribution function $f(\mathbf{r},\mathbf{v},t)$ is solution of the Vlasov equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}-\nabla\Phi\frac{\partial f}{\partial \mathbf{v}}=0, \tag{14}$$

where $\Phi(\mathbf{r},t)=\int u(|\mathbf{r}-\mathbf{r}'|)f(\mathbf{r}',\mathbf{v}',t)d\mathbf{r}'d\mathbf{v}'$ is the smooth potential created by f. The Vlasov equation describes the collisionless evolution of the system due to mean field effects only, before the cumulative nature of the collisions becomes manifest on a timescale $t_{coll}\sim Nt_D$ or larger. Starting from an initial condition which is dynamically unstable, the Vlasov equation coupled to a long-range potential of interaction develops an intricate filamentation in phase space at smaller and smaller scales. In this sense, the fine-grained distribution function $f(\mathbf{r},\mathbf{v},t)$ never achieves equilibrium. However, if we locally average over the filaments, the resulting "coarse-grained" distribution function $\overline{f}(\mathbf{r},\mathbf{v},t)$ will achieve a steady state on a timescale $\sim t_D$. Since the Vlasov equation is only valid in the collisionless regime $t\ll t_{coll}$, this corresponds to a quasi-stationary state (QSS) that will slowly evolve under the effect of collisions on a timescale $\sim Nt_D$ or larger. We can try to predict this QSS in terms of a statistical mechanics of the Vlasov equation, using the approach of Lynden-Bell [20] developed for collisionless stellar systems (see also [21]). In the case where the fine-grained distribution function $f(\mathbf{r},\mathbf{v},t)$ takes only two values 0 and η_0, the statistical equilibrium state maximizes the Lynden-Bell entropy

$$S_{L.B.}=-\int\left\{\frac{\overline{f}}{\eta_0}\ln\frac{\overline{f}}{\eta_0}+\left(1-\frac{\overline{f}}{\eta_0}\right)\ln\left(1-\frac{\overline{f}}{\eta_0}\right)\right\}d\mathbf{r}d\mathbf{v}, \tag{15}$$

at fixed mass and energy. This leads to the coarse-grained distribution function

$$\overline{f}=\frac{\eta_0}{1+e^{\beta\eta_0(\frac{v^2}{2}+\Phi)-\mu}}. \tag{16}$$

Note that the mixing entropy (15) is formally similar to the Fermi-Dirac entropy and the equilibrium distribution (16) is formally similar to the Fermi-Dirac distribution. An effective "exclusion principle", similar to the Pauli principle in quantum mechanics, arises in the theory of violent relaxation because the different phase levels cannot overlap. We stress that the Lynden-Bell theory is based on an assumption of ergodicity. Indeed, it implicity assumes that

the phase elements mix efficiently during the dynamics so that the QSS is the *most mixed state* compatible with the integral constraints of the Vlasov equation. This may not always be the case as discussed in the sequel.

We can try to determine the dynamical equation satisfied by the coarse-grained distribution function $\overline{f}(\mathbf{r}, \mathbf{v}, t)$ by developing a quasilinear theory of the Vlasov equation. We decompose the distribution function in the form $f = \overline{f} + \tilde{f}$ where $\overline{f}$ is the coarse-grained distribution function and $\tilde{f} \ll \overline{f}$ a fluctuation around it. Substituting this decomposition in Eq. (14) and taking the local average, we get

$$\frac{\partial \overline{f}}{\partial t} + L\overline{f} = \frac{\partial}{\partial \mathbf{v}} \overline{\nabla \tilde{\Phi} \tilde{f}}, \tag{17}$$

where $L = \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \nabla \Phi \frac{\partial}{\partial \mathbf{v}}$ is an advection operator in phase space constructed with the smooth field. Subtracting Eq. (17) from Eq. (14) and neglecting nonlinear terms in the fluctuations, we obtain an equation for the perturbation

$$\frac{\partial \tilde{f}}{\partial t} + L\tilde{f} = \nabla \tilde{\Phi} \frac{\partial \overline{f}}{\partial \mathbf{v}}. \tag{18}$$

Equations (17) and (18) are formally similar to Eqs. (3) and (4) of the previous section but with a completely different interpretation. In Sec. 2.1, the subdynamics was played by f_d (a sum of δ-functions) and the macrodynamics by f (a smooth field). The smooth field averages over the positions of the δ-functions that strongly fluctuate. In the phase of violent relaxation, the "smooth" field f develops itself a finely striated structure and strongly fluctuates. Therefore, it is *not* smooth at a higher scale of resolution and a second smoothing procedure (coarse-graining) must be introduced. In that case, the subdynamics is played by f and the macrodynamics by $\overline{f}$. The coarse-grained field averages over the positions of the filaments.

The coupled equations (17) and (18) can be solved by an iterative procedure similar to that developed in Sec. 2.1 and we finally obtain

$$\begin{aligned}\frac{\partial \overline{f}}{\partial t} + L\overline{f} = \frac{\partial}{\partial v^\mu} \int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 d\mathbf{r}_2 d\mathbf{v}_2 \frac{F^\mu}{m}(1 \to 0) G_1(t, t-\tau) G(t, t-\tau) \\ \times \Big\{ \frac{F^\nu}{m}(2 \to 0) \overline{\tilde{f}(\mathbf{r}_1, \mathbf{v}_1, t-\tau)\tilde{f}(\mathbf{r}_2, \mathbf{v}_2, t-\tau)} \frac{\partial \overline{f}}{\partial v^\nu}(\mathbf{r}, \mathbf{v}, t-\tau) \\ + \frac{F^\nu}{m}(2 \to 1) \overline{\tilde{f}(\mathbf{r}, \mathbf{v}, t-\tau)\tilde{f}(\mathbf{r}_2, \mathbf{v}_2, t-\tau)} \frac{\partial \overline{f}}{\partial v_1^\nu}(\mathbf{r}_1, \mathbf{v}_1, t-\tau) \Big\}.\end{aligned} \tag{19}$$

To close the system, it remains for one to evaluate the correlation function $\overline{\tilde{f}(\mathbf{r}, \mathbf{v}, t)\tilde{f}(\mathbf{r}_1, \mathbf{v}_1, t)}$. We shall assume that the mixing in phase space is sufficiently efficient so that the scale of the kinematic correlations is small with respect to the coarse-graining mesh size. In that case,

$$\overline{\tilde{f}(\mathbf{r}, \mathbf{v}, t)\tilde{f}(\mathbf{r}_1, \mathbf{v}_1, t)} = \epsilon_r^d \epsilon_v^d \delta(\mathbf{r} - \mathbf{r}_1)\delta(\mathbf{v} - \mathbf{v}_1)\overline{\tilde{f}^2}(\mathbf{r}, \mathbf{v}, t), \tag{20}$$

where ϵ_r and ϵ_v are the resolution scales in position and velocity respectively. Now,

$$\overline{\tilde{f}^2} = \overline{(f - \overline{f})^2} = \overline{f^2} - \overline{f}^2. \tag{21}$$

We shall assume, for simplicity, that the initial condition in phase space consists of patches where the distribution function takes a unique value $f = \eta_0$ surrounded by vacuum ($f = 0$). In this two-levels approximation $\overline{f^2} = \overline{\eta_0 \times f} = \eta_0 \overline{f}$ and, therefore,

$$\overline{\tilde{f}(\mathbf{r}, \mathbf{v}, t)\tilde{f}(\mathbf{r}_1, \mathbf{v}_1, t)} = \epsilon_r^d \epsilon_v^d \delta(\mathbf{r} - \mathbf{r}_1)\delta(\mathbf{v} - \mathbf{v}_1)\overline{f}(\eta_0 - \overline{f}). \tag{22}$$

Substituting this expression in Eq. (19) and carrying out the integrations on $\mathbf{r}_2$ and $\mathbf{v}_2$, we obtain

$$\frac{\partial \overline{f}}{\partial t} + L\overline{f} = \epsilon_r^d \epsilon_v^d \frac{\partial}{\partial v^\mu} \int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 \frac{F^\mu}{m}(1 \to 0) G(t, t-\tau) \frac{F^\nu}{m}(1 \to 0)$$
$$\times \left\{ \overline{f}_1(\eta_0 - \overline{f}_1) \frac{\partial \overline{f}}{\partial v^\nu} - \overline{f}(\eta_0 - \overline{f}) \frac{\partial \overline{f}_1}{\partial v_1^\nu} \right\}_{t-\tau}, \tag{23}$$

where $f = f(\mathbf{r}, \mathbf{v}, t-\tau)$ and $f_1 = f(\mathbf{r}_1, \mathbf{v}_1, t-\tau)$. This equation is expected to describe the late quiescent stages of the relaxation process when the fluctuations have weaken so that the quasilinear approximation can be implemented. It does not describe the early very chaotic process of violent relaxation driven by the strong fluctuations of the potential. The quasilinear theory of the Vlasov equation is therefore a theory of "quiescent" collisionless relaxation.

Equation (23) is similar, in structure, to Eq. (13) describing the collisional evolution of the system with, nevertheless, three important differences: (i) the fluctuating force $\mathcal{F}(1 \to 0)$ is replaced by the direct force $F(1 \to 0)$ because the fluctuations are taken into account differently. (ii) The distribution function f in the collisional term of Eq. (13) is replaced by the product $\overline{f}(\eta_0 - \overline{f})$ in Eq. (23). This nonlinear term arises from the effective "exclusion principle", discovered by Lynden-Bell, accounting for the non-overlapping of phase levels in the collisionless regime. This is consistent with the Fermi-Dirac-like entropy (15) and Fermi-Dirac-like distribution (16) at statistical equilibrium (iii) Considering the dilute limit $\overline{f} \ll \eta_0$ to fix the ideas, we see that the equations (23) and (13) have the same mathematical form differing only in the prefactors: the mass m of a particle in Eq. (13) is replaced by the mass $\eta_0 \epsilon_r^d \epsilon_v^d$ of a completely filled macrocell in Eq. (23). This implies that the timescales of collisional and collisionless relaxation are in the ratio

$$\frac{t_{ncoll}}{t_{coll}} \sim \frac{m}{\eta_0 \epsilon_r^d \epsilon_v^d}. \tag{24}$$

Since $\eta_0 \epsilon_r^d \epsilon_v^d \gg m$, this ratio is in general quite small implying that the collisionless relaxation is much more rapid than the collisional relaxation. Typically, t_{ncoll} is of the order of a few dynamical times t_D (its precise value depends on the size of the mesh) while t_{coll} is of order $\sim N t_D$ or larger. The kinetic equation (23) conserves the mass and, presumably, the energy. By contrast, we cannot prove an H-theorem for the Lynden-Bell entropy (15). Indeed, the time variation of the Lynden-Bell entropy is of the form

$$\dot{S}_{L.B.} = \frac{1}{2} \epsilon_r^d \epsilon_v^d \int d\mathbf{r} d\mathbf{v} d\mathbf{r}_1 d\mathbf{v}_1 \frac{1}{\overline{f}(\eta_0 - \overline{f}) \overline{f}_1 (\eta_0 - \overline{f}_1)} \int_0^t d\tau Q(t) G(t, t-\tau) Q(t-\tau), \tag{25}$$

$$Q(t) = \frac{F^\mu}{m}(1 \to 0, t) \left[\overline{f}_1(\eta_0 - \overline{f}_1) \frac{\partial \overline{f}}{\partial v^\mu} - \overline{f}(\eta_0 - \overline{f}) \frac{\partial \overline{f}_1}{\partial v_1^\mu} \right], \tag{26}$$

and its sign is not necessarily positive. This depends on the importance of memory terms. In addition, even if Eq. (23) conserves energy and increases the Fermi-Dirac entropy monotonically, this does not necessarily imply that the system will converge towards the Lynden-Bell distribution (16). It has been observed in several experiments and numerical simulations that the QSS does not coincide with the statistical equilibrium state predicted by Lynden-Bell. This *incomplete relaxation* [22] is usually explained by a lack of ergodicity and "incomplete mixing". In fact, very few is known concerning kinetic equations of the form of Eq. (23) and it is not clear whether the Lynden-Bell distribution (16) is a stationary solution of that equation (and

if it is the only one). As explained in Paper III for the kinetic equation (13) describing the collisional relaxation, the relaxation may stop because the current $\mathbf{J}$ vanishes due to the *absence of resonances*. This argument may also apply to Eq. (23) which has a similar structure and can be a cause for incomplete relaxation. The system tries to approach the statistical equilibrium state (as indicated by the increase of the entropy) but may be trapped in a QSS that is different from the statistical prediction (16). This QSS is a steady solution of Eq. (23), or more generally (19), which cancels individually the advective term (l.h.s.) and the effective collision term (r.h.s.). This determines a subclass of steady states of the Vlasov equation (cancellation of the l.h.s.) such that the complicated "turbulent" current $\mathbf{J}$ in the r.h.s. vanishes. This offers a large class of possible steady state solutions that can explain the deviation between the QSS and the Lynden-Bell statistical equilibrium state (16) observed, in certain cases, in simulations and experiments of violent relaxation. Other causes of incomplete relaxation, due to the rapid decay of the fluctuations in space and time (leading to a small value of the current), will be described in Sec. 2.3.

2.3 The case of stellar systems

The case of stellar systems is special and deserves a specific discussion. These systems are spatially inhomogeneous but, due to the divergence of the gravitational force $\mathbf{F}(1 \to 0)$ when $\mathbf{r}_1 \to \mathbf{r}$, the integral in the r.h.s. of Eq. (23) can be evaluated by making a *local approximation* which amounts to replacing $f(\mathbf{r}_1, \mathbf{v}_1, t)$ by $f(\mathbf{r}, \mathbf{v}_1, t)$. This approximation is justified by the fact that the diffusion coefficient diverges logarithmically when $\mathbf{r}_1 \to \mathbf{r}$ (see below). We shall also make a markovian approximation $f(\mathbf{r}, \mathbf{v}_1, t-\tau) \simeq f(\mathbf{r}, \mathbf{v}_1, t)$, $f(\mathbf{r}, \mathbf{v}, t-\tau) \simeq f(\mathbf{r}, \mathbf{v}, t)$ and extend the time integration to $+\infty$. Then, Eq. (23) becomes

$$\frac{\partial \overline{f}}{\partial t} + L\overline{f} = \epsilon_r^3 \epsilon_v^3 \frac{\partial}{\partial v^\mu} \int_0^{+\infty} d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 \frac{F^\mu}{m}(1 \to 0, t) \frac{F^\nu}{m}(1 \to 0, t-\tau)$$
$$\times \left\{ \overline{f}_1(\eta_0 - \overline{f}_1) \frac{\partial \overline{f}}{\partial v^\nu} - \overline{f}(\eta_0 - \overline{f}) \frac{\partial \overline{f}_1}{\partial v_1^\nu} \right\}, \quad (27)$$

where now $f = f(\mathbf{r}, \mathbf{v}, t)$ and $f_1 = f(\mathbf{r}, \mathbf{v}_1, t)$. Making a linear trajectory approximation $\mathbf{v}_i(t-\tau) = \mathbf{v}_i(t)$ and $\mathbf{r}_i(t-\tau) = \mathbf{r}_i - \mathbf{v}_i \tau$, we can perform the integrations on $\mathbf{r}_1$ and τ like in Appendix A of Paper II. This yields the generalized Landau equation

$$\frac{\partial \overline{f}}{\partial t} + L\overline{f} = \pi (2\pi)^3 \epsilon_r^3 \epsilon_v^3 \frac{\partial}{\partial v^\mu} \int d\mathbf{v}_1 d\mathbf{k} k^\mu k^\nu \hat{u}(k)^2 \delta(\mathbf{k}\cdot\mathbf{w}) \left\{ \overline{f}_1(\eta_0 - \overline{f}_1) \frac{\partial \overline{f}}{\partial v^\nu} - \overline{f}(\eta_0 - \overline{f}) \frac{\partial \overline{f}_1}{\partial v_1^\nu} \right\}. \quad (28)$$

As a result of the local approximation, the effect of the spatial inhomogeneity is only retained in the advective term L in the l.h.s. of Eq. (28). The same approximations are made for collisional stellar systems leading to Eq. (II-44) of Paper II. Equation (28) can also be written in the form

$$\frac{\partial \overline{f}}{\partial t} + L\overline{f} = 2\pi G^2 \epsilon_r^3 \epsilon_v^3 \ln\Lambda \frac{\partial}{\partial v^\mu} \int d\mathbf{v}_1 \frac{w^2 \delta^{\mu\nu} - w^\mu w^\nu}{w^3} \left\{ \overline{f}_1(\eta_0 - \overline{f}_1) \frac{\partial \overline{f}}{\partial v^\nu} - \overline{f}(\eta_0 - \overline{f}) \frac{\partial \overline{f}_1}{\partial v_1^\nu} \right\}, \quad (29)$$

where $\ln\Lambda = \int_0^{+\infty} dk/k$ is the Coulombian factor. It exibits a logarithmic divergence at small and large scales and it must be regularized by introducing some cut-offs, writing $\ln\Lambda = \ln(L_{max}/L_{min})$. The integral at large scales must be cut-off at the system size $L_{max} \sim R$ (or Jeans length) which plays the role of the Debye length in the present context (see Paper III).

For collisional stellar systems, the integral at small scales must be cut-off at the Landau length $L_{min} \sim Gm/v_{typ}^2$ with $v_{typ}^2 \sim GM/R$ corresponding to a deflexion at 90^o of the particles' trajectory. This yields a Coulomb factor $\ln\Lambda \sim \ln N$. In the present context, the integral at small scales must be cut-off at the resolution length ϵ_r. Therefore, $\ln\Lambda = \ln(R/\epsilon_r)$. This implies that the timescale of collisional relaxation and the timescale of violent relaxation are in the ratio

$$\frac{t_{ncoll}}{t_{coll}} \sim \frac{m}{\eta_0 \epsilon_r^3 \epsilon_v^3} \frac{\ln(R/\epsilon_r)}{\ln N}. \tag{30}$$

It is easy to check [25] that Eq. (29) conserves the mass and the energy, that it monotonically increases the Lynden-Bell entropy (15) (H-theorem) and that its only stationary solution is the Lynden-Bell distribution (16). Therefore, the kinetic equation (29) tends to reach the Lynden-Bell distribution (16). However, there are several reasons why it cannot attain it: (i) *Evaporation:* for self-gravitating systems, it is well-known that the Lynden-Bell distribution (16) coupled to the Poisson equation has infinite mass so that there is no physical distribution of the form (16) in an infinite domain. The system can increase the Lynden-Bell entropy indefinitely by evaporating. Therefore, the generalized Vlasov-Landau equation (29) has no steady state with finite mass and the distribution function tends to spreads indefinitely. (ii) *Incomplete relaxation in space:* The turbulent current $\mathbf{J}$ in Eq. (29), or more generally in Eq. (19), is driven by the fluctuations $f_2 \equiv \overline{\tilde{f}^2}$ of the distribution function (generating the fluctuations $\delta\Phi$ of the potential). In the "mixing region" of phase space where the fluctuations are strong, the DF tends to reach the Lynden-Bell distribution (16). As we depart from the "mixing region", the fluctuations decay ($f_2 \to 0$) and the mixing is less and less efficient $\|\mathbf{J}\| \to 0$. In these regions, the system takes a long time to reach the Lynden-Bell distribution (16) and, in practice, cannot attain it in the time available (see (iii)). In the two levels case, we have $f_2 = \overline{f}(\eta_0 - \overline{f})$. Therefore, the phase space regions where $\overline{f} \to 0$ or $\overline{f} \to \eta_0$ do not mix well (the diffusion current $\mathbf{J}$ is weak) and the observed DF can be sensibly different from the Lynden-Bell distribution in these regions of phase space. This concerns essentially the core ($\overline{f} \to \eta_0$) and the tail ($\overline{f} \to 0$) of the distribution. (iii) *Incomplete relaxation in time:* during violent relaxation, the system tends to approach the statistical equilibrium state (16). However, as it approaches equilibrium, the fluctuations of the gravitational field, which are the engine of the evolution, become less and less effective to drive the relaxation. This is because the scale of the fluctuations becomes smaller and smaller as time goes on. This effect can be taken into account in the kinetic theory by considering that the correlation lengths $\epsilon_r(t)$ and $\epsilon_v(t)$ decrease with time so that, in the kinetic equation (29), the prefactor $\epsilon_r(t)\epsilon_v(t) \to 0$ for $t \to +\infty$. As a result, the "turbulent" current $\mathbf{J}$ in Eq. (29) can vanish *before* the system has reached the statistical equilibrium state (16). In that case, the system can be trapped in a QSS that is a steady solution of the Vlasov equation different from the statistical prediction (16). Similar arguments have been given in [24] on the basis of a more phenomenological kinetic theory of violent relaxation. On longer timescale, the encounters must be taken into account. Then the system is described by a collisional kinetic Vlasov-Landau equation of the form (III-41). This equation conserves the mass, the energy (kinetic + potential) and monotonically increases the Boltzmann entropy. The mean field Maxwell-Boltzmann distribution (I-24) is the only stationary solution of this equation so that the system tends to reach this distribution on a timescale $(N/\ln N)t_D$. In practice, however, the convergence to the Boltzmann distribution is hampered by the *escape of stars* and by the *gravothermal catastrophe* [5, 6, 7].

2.4 Physical interpretation of the QSS

Based on the preceding kinetic theory, we propose the following interpretation [10] of the QSS observed in Hamiltonian systems with long-range interactions:

1. The QSS results from a process of phase mixing and violent relaxation. This is a purely collisionless process driven by the fluctuations of the mean-field potential. It takes place on a timescale of a few dynamical times where the Vlasov equation is valid. The QSS is a nonlinearly dynamically stable stationary solution of the Vlasov equation on the coarse-grained scale, i.e. the *coarse-grained* DF $\overline{f}_{QSS}(\mathbf{r}, \mathbf{v})$ is a stable stationary solution of the Vlasov equation. Since the Vlasov equation admits an infinite number of stationary solutions, it is not easy to predict the one which will be dynamically selected by the process of violent relaxation.

2. In principle, the distribution $\overline{f}_{QSS}(\mathbf{r}, \mathbf{v})$ of the QSS can be predicted from the statistical theory of the Vlasov equation developed by Lynden-Bell [20]. The distribution $\overline{f}_{L.B.}(\mathbf{r}, \mathbf{v})$ depends on the details of the initial condition (in addition to the value of the mass and the energy) because of the conservation of the Casimir constraints [21]. The coarse-grained DF predicted by Lynden-Bell looks like a sort of superstatistics.

3. In many cases, the prediction of Lynden-Bell works well [26, 27, 28, 29]. In certain cases, the prediction of Lynden-Bell fails because of the complicated problem of *incomplete relaxation* [22]. The system tends to reach the Lynden-Bell distribution (as implied by the increase of the Lynden-Bell entropy) but cannot attain it because the fluctuations of the potential (which drive the evolution) fade away before the system has reached the most mixed state. Therefore, the incompleteness of the violent relaxation is of dynamical origin. In such cases, the QSS can take forms that are different from the statistical prediction, i.e. $\overline{f}_{QSS} \neq \overline{f}_{L.B.}$. Thus, other distributions, that are stable stationary solutions of the Vlasov equation, can emerge. For example, the Tsallis distributions [30] are *particular* stationary solutions of the Vlasov equation (polytropes) [31] that can sometimes be reached as a result of an incomplete violent relaxation. Several examples have been exhibited where the QSS [32, 33, 34, 35] or the transient stages of the collisional relaxation [36, 37, 35] are remarkably well fitted by Tsallis distributions (see the detailed discussion of Paper III [4]). This suggests that Tsallis distributions may represent "attractors" of the Vlasov equation in case of incomplete relaxation, for some particular initial conditions. However, they are not "universal attractors" [4]. Indeed, other distributions have been observed that differ both from the Lynden-Bell and the Tsallis distributions. This is clear for galaxies in astrophysics that are neither isothermal nor polytropic [40]. There are also cases where the system does not reach a QSS and develops instead long-lasting oscillations [41, 42]. It would be interesting to know whether these different possible behaviours are captured by the kinetic equation (23).

4. Since the Lynden-Bell/Vlasov approach is restricted to the Boltzmann μ-space, that is only a projection of the full Gibbs Γ-space, one could fear that some fundamental properties of the latter (fractal structures, etc...) could be lost in that approach. In fact, we believe that the Vlasov equation correctly describes the regime where the QSS appears. Therefore, in this regime, all the physics of the problem is contained in the Vlasov equation evolving in μ-space. However, the Vlasov equations is a very complicated equation (like the Euler equations of turbulence for example). In particular, it can exhibit fractal structures and non-ergodic behaviours just as the N-body system does. Therefore, the Vlasov equation is not in contradiction with a complex structure of phase space: the striking features that have been observed for the N-body problem such as QSS [34, 37, 29, 35], phase-space holes/clumps [41, 42],

[4] Tsallis entropies apply when the phase space of a system is fractal or multi-fractal. The fractal properties of the process of violent relaxation are not known. For the HMF model, an interesting regime where Tsallis thermodynamics seems to apply [38] has been found above a critical magnetization [39].

anomalous diffusion [38], non-ergodic behaviours etc. should also be observed with the Vlasov equation (except if they are due to finite N-effects which is also a possibility to consider).

3 Relaxation of a test particle in a bath

In this section, we study the relaxation of a test particle in a bath of field particles. Specifically, we consider a collection of N particles at statistical equilibrium (thermal bath) and introduce a new particle in the system. To leading order in $N \to +\infty$, the particle is advected by the mean flow in phase space. However, due to finite N effects (graininess), the test particle undergoes discrete interactions with the particles of the bath and progressively acquires their distribution. We wish to study this stochastic process. The probability density $P(\mathbf{r}, \mathbf{v}, t)$ of finding the test particle in $\mathbf{r}$ with velocity $\mathbf{v}$ at time t is governed by a Fokker-Planck equation involving a term of diffusion and a term of friction. These results are well-known when the system is spatially homogeneous and memory effects can be neglected, as in the case of plasma physics. In the present work, we shall develop a method that allows to treat spatially inhomogeneous systems and that takes into account non-markovian effects. Our approach is also valid if the bath is made of an out-of-equilibrium distribution of field particles that evolves *slowly* so that it can be assumed stationary on a timescale Nt_D, which is the typical relaxation time of the test particle in the bath. This is the case in particular for one dimensional systems for which the Lenard-Balescu collision term vanishes at order $O(1/N)$. Therefore, any stable steady solution of the Vlasov equation does not evolve on a timescale Nt_D [43, 16, 4].

3.1 Diffusion coefficient

The increment of the velocity of the test particle between $t-s$ and t due to the fluctuations of the force is

$$\Delta v^{\mu} = \int_{t-s}^{t} \mathcal{F}^{\mu}(t')dt'. \tag{31}$$

After standard calculations (see, e.g., Sec. 4.2 of [44]), the second moment of the velocity increment can be written

$$\left\langle \frac{\Delta v^{\mu}\Delta v^{\nu}}{2s} \right\rangle = \frac{1}{s}\int_{0}^{s} (s+\tau)\langle \mathcal{F}^{\mu}(t)\mathcal{F}^{\nu}(t-\tau)\rangle d\tau. \tag{32}$$

We shall assume that the correlation function of the force decreases more rapidly than τ^{-1} (note, parenthetically, that this is not the case for the correlation function of the gravitational force which precisely decreases as τ^{-1} [45]). Then, taking the limit $s \to +\infty$, we find that the diffusion coefficient is given by the Kubo formula

$$D^{\mu\nu} = \left\langle \frac{\Delta v^{\mu}\Delta v^{\nu}}{2\Delta t} \right\rangle \equiv \lim_{s\to +\infty} \left\langle \frac{\Delta v^{\mu}\Delta v^{\nu}}{2s} \right\rangle = \int_{0}^{+\infty} \langle \mathcal{F}^{\mu}(t)\mathcal{F}^{\nu}(t-\tau)\rangle d\tau. \tag{33}$$

On the other hand, after straightforward calculations (see, e.g., Sec. 4.1 of [44]), we obtain

$$\begin{aligned} \langle \mathcal{F}^{\mu}(t)\mathcal{F}^{\nu}(t-\tau)\rangle &= N\langle \mathcal{F}^{\mu}(1\to 0,t)\mathcal{F}^{\nu}(1\to 0,t-\tau)\rangle \\ &= \int d\mathbf{r}_1 d\mathbf{v}_1 \mathcal{F}^{\mu}(1\to 0,t)\mathcal{F}^{\nu}(1\to 0,t-\tau)\frac{f}{m}(\mathbf{r}_1,\mathbf{v}_1). \end{aligned} \tag{34}$$

Therefore, combining Eqs. (33) and (34), we get

$$D^{\mu\nu} = \int_0^{+\infty} d\tau d\mathbf{r}_1 d\mathbf{v}_1 \mathcal{F}^\mu(1\to 0,t)\mathcal{F}^\nu(1\to 0,t-\tau)\frac{f}{m}(\mathbf{r}_1,\mathbf{v}_1). \tag{35}$$

For a spatially homogeneous distribution, the diffusion coefficient reduces to

$$D^{\mu\nu} = \int_0^{+\infty} d\tau d\mathbf{r}_1 d\mathbf{v}_1 F^\mu(1\to 0,t)F^\nu(1\to 0,t-\tau)\frac{f}{m}(\mathbf{v}_1). \tag{36}$$

If we neglect collective effects, the force (by unit of mass) created by the field particle 1 on the test particle 0 can be written (see Paper I):

$$\mathbf{F}(1\to 0,t) = -im\int \mathbf{k}\hat{u}(k)e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_1)}d\mathbf{k}. \tag{37}$$

At time $t-\tau$, we have

$$\mathbf{F}(1\to 0,t-\tau) = -im\int \mathbf{k}\hat{u}(k)e^{i\mathbf{k}(\mathbf{r}(t-\tau)-\mathbf{r}_1(t-\tau))}d\mathbf{k}. \tag{38}$$

To leading order in $N\to+\infty$, the particles follow rectilinear trajectories so that $\mathbf{r}_i(t-\tau) = \mathbf{r}_i - \mathbf{v}_i\tau$ where $\mathbf{r}_i = \mathbf{r}_i(t)$ and $\mathbf{v}_i = \mathbf{v}_i(t)$ denote their position and velocity at time t. Then, we get (with $\mathbf{x} = \mathbf{r}-\mathbf{r}_1$ and $\mathbf{w} = \mathbf{v}-\mathbf{v}_1$):

$$\mathbf{F}(1\to 0,t-\tau) = -im\int \mathbf{k}\hat{u}(k)e^{i\mathbf{k}(\mathbf{x}-\mathbf{w}\tau)}d\mathbf{k}. \tag{39}$$

Substituting this expression in Eq. (36) and carrying the integrations on $\mathbf{r}_1$ and τ, we obtain after straightforward calculations

$$D^{\mu\nu} = \pi(2\pi)^d m\int k^\mu k^\nu \hat{u}(k)^2\delta(\mathbf{k}\cdot\mathbf{w})f(\mathbf{v}_1)d\mathbf{k}d\mathbf{v}_1. \tag{40}$$

If we take into account collective effects (see Appendix B), we have to replace Eq. (39) by

$$\mathbf{F}(1\to 0,t-\tau) = -im\int \mathbf{k}\frac{\hat{u}(k)}{\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_1)}e^{i\mathbf{k}(\mathbf{x}-\mathbf{w}\tau)}d\mathbf{k}. \tag{41}$$

Then, we get

$$D^{\mu\nu} = \pi(2\pi)^d m\int k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}\delta(\mathbf{k}\cdot\mathbf{w})f(\mathbf{v}_1)d\mathbf{k}d\mathbf{v}_1. \tag{42}$$

The calculation of the diffusion coefficient tensor $D^{\mu\nu}$ for different potentials of interaction and different dimensions of space is performed in Paper II and in [16]. For one dimensional systems, we have the simple result

$$D(v) = 4\pi^2 mf(v)\int_0^{+\infty} \frac{k\hat{u}(k)^2}{|\epsilon(k,kv)|^2}dk, \tag{43}$$

where we have used $\delta(k(v-v_1)) = (1/|k|)\delta(v-v_1)$ to perform the integration on v_1.

3.2 Friction coefficient

In addition to its diffusive motion, a test particle evolving in a bath of field particles undergoes a dynamical friction. The friction corresponds to the response of the field particles to the perturbation caused by the test particle, as in a polarization process. The test particle modifies the distribution of the field particles and the retroaction of this perturbation on the test particle creates a friction. The expression of the friction force can be derived from a linear response theory starting from the Liouville equation as done in Kandrup [46]. In this section, we show that it can also be obtained from the Klimontovich equation. This will make a close connection to the quasilinear theory developed in Sec. 2.

The introduction of a test particle in a bath of field particles modifies the distribution function $f(\mathbf{r},\mathbf{v},t)$ of the bath. Since this perturbation is small, it can be described by the linearized equation

$$\frac{\partial \delta f}{\partial t}+L\delta f=\nabla\delta\Phi\frac{\partial f}{\partial \mathbf{v}}, \tag{44}$$

whose formal solution is

$$\delta f(t)=\int_0^t d\tau G(t,t-\tau)\nabla\delta\Phi(t-\tau)\frac{\partial f}{\partial \mathbf{v}}(t-\tau). \tag{45}$$

We have used the fact that, initially, $\delta f(0)=0$. On the other hand, the perturbation of the force in $\mathbf{r}$ is given by

$$-\nabla\delta\Phi(\mathbf{r},t)=\frac{1}{m}\int \mathbf{F}(1\rightarrow 0)\delta f_1(t)d\mathbf{x}_1+\int \mathcal{F}(1\rightarrow 0)\delta(\mathbf{r}_1-\mathbf{r}_P(t))d\mathbf{r}_1, \tag{46}$$

where $\mathbf{r}_P(t)$ denotes the position of the test particle. The second term is the force created by the test particle and the first term is the fluctuation of the force due to the perturbed density distribution of the field particles. Substituting Eq. (45) in Eq. (46) we obtain

$$-\nabla\delta\Phi(\mathbf{r},t)=\frac{1}{m}\int_0^t d\tau\int d\mathbf{x}_1\mathbf{F}(1\rightarrow 0)G_1(t,t-\tau)\frac{\partial \delta\Phi_1}{\partial r_1^{\nu}}(t-\tau)\frac{\partial f_1}{\partial v_1^{\nu}}(t-\tau)$$
$$+\int \mathcal{F}(1\rightarrow 0)\delta(\mathbf{r}_1-\mathbf{r}_P(t))d\mathbf{r}_1. \tag{47}$$

This is an integral equation for $-\nabla\delta\Phi(\mathbf{r},t)$. For a spatially homogeneous system, one can solve this equation exactly by using Laplace-Fourier transforms. This is how the dielectric function enters in the problem (see Appendix B). In order to treat more general systems that are not necessarily homogeneous, we shall make an approximation which amounts to neglecting some collective effects. We solve Eq. (47) by an iterative process: we first neglect the first term in the r.h.s. of Eq. (47) keeping only the contribution of the test particle. Then, we substitute this value in the first term of the r.h.s of Eq. (47). This operation gives

$$-\nabla\delta\Phi(\mathbf{r},t)=-\frac{1}{m}\int_0^t d\tau\int d\mathbf{x}_1 d\mathbf{r}_2\mathbf{F}(1\rightarrow 0)G_1(t,t-\tau)\mathcal{F}^{\nu}(2\rightarrow 1)$$
$$\times\frac{\partial f_1}{\partial v_1^{\nu}}(t-\tau)\delta(\mathbf{r}_2-\mathbf{r}_P(t-\tau))+\int \mathcal{F}(1\rightarrow 0)\delta(\mathbf{r}_1-\mathbf{r}_P(t))d\mathbf{r}_1. \tag{48}$$

This quantity represents the fluctuation of the field in $\mathbf{r}$ caused by the introduction of the test particle in the system and taking into account of the retroaction of the field particles.

If we evaluate this expression at the position $\mathbf{r}_P$ of the test particle and subtract the second term (self-interaction), we obtain the friction force felt by the test particle in response to the perturbation that it caused. Denoting now by 0 the position of the test particle, we find that the friction is given by

$$F^{\mu}_{pol} = -\frac{1}{m}\int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)\mathcal{F}^{\nu}(0\to 1,t-\tau)\frac{\partial f}{\partial v^{\nu}}(\mathbf{r}_1(t-\tau),\mathbf{v}_1(t-\tau)). \qquad (49)$$

For a thermal bath, where the distribution of the field particles is given by $f(\mathbf{r}_1,\mathbf{v}_1) = Ae^{-\beta m(v_1^2/2+\Phi(\mathbf{r}_1))}$, we obtain

$$F^{\mu}_{pol} = \beta\int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)\mathcal{F}(0\to 1,t-\tau)\cdot\mathbf{v}_1(t-\tau)f(\mathbf{r}_1,\mathbf{v}_1), \qquad (50)$$

where we have used $f(\mathbf{r}_1(t-\tau),\mathbf{v}_1(t-\tau)) = f(\mathbf{r}_1(t),\mathbf{v}_1(t))$ since f is a stationary solution of the Vlasov equation. This is equivalent to the result of Kandrup [46] based on the Liouville equation but it is obtained here in a simpler manner from the Klimontovich equation. We can also obtain this result in a slightly different way. We approximate $-\nabla\delta\Phi(\mathbf{r},t)$ in Eq. (44) by the force $\mathcal{F}(P\to 0)$ created by the test particle only so that

$$\frac{\partial\delta f}{\partial t} + L\delta f = -\mathcal{F}(P\to 0)\frac{\partial f}{\partial\mathbf{v}}. \qquad (51)$$

This equation can be solved with a Green function yielding

$$\delta f(t) = -\int_0^t d\tau G(t,t-\tau)\mathcal{F}(P\to 0,t-\tau)\frac{\partial f}{\partial\mathbf{v}}(t-\tau). \qquad (52)$$

This represents the perturbation of the distribution function of the field particles caused by the introduction of a test particle in the system. This perturbation produces in turn a force which acts as a friction on the test particle (by retroaction). If we substitute Eq. (52) in the first part of Eq. (46) and evaluate this quantity at the position of the test particle, we recover Eq. (49) for the friction.

If we now consider a spatially homogeneous distribution of field particles, the expression of the friction force becomes

$$F^{\mu}_{pol} = \frac{1}{m}\int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\frac{\partial f}{\partial v^{\nu}}(\mathbf{v}_1), \qquad (53)$$

where we have used $\mathbf{v}_1(t-\tau) = \mathbf{v}_1(t)$ to leading order in $N\to+\infty$. Taking the limit $t\to+\infty$, we get

$$F^{\mu}_{pol} = \frac{1}{m}\int_0^{+\infty} d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\frac{\partial f}{\partial v^{\nu}}(\mathbf{v}_1). \qquad (54)$$

This is a sort of generalized Kubo relation involving the gradient of the distribution function in velocity space instead of the distribution function itself. *The nice similarity in the expressions of the diffusion coefficient (36) and friction force (54) is worth mentioning.* The integrals on $\mathbf{r}_1$ and τ can be calculated in the same manner as in Sec. 3.1 and we obtain

$$F^{\mu}_{pol} = \pi(2\pi)^d m\int d\mathbf{v}_1 d\mathbf{k}\hat{u}(k)^2 k^{\mu}k^{\nu}\delta(\mathbf{k}\cdot\mathbf{w})\frac{\partial f_1}{\partial v_1^{\nu}}. \qquad (55)$$

In order to take into account collective effects, we can follow the approach of Hubbard [47]. The force (by unit of mass) created in $\mathbf{r}$ by the introduction of the test particle is

$$\mathbf{F}(P \to 0) = -im \int \mathbf{k} \frac{\hat{u}(k)}{\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_P)} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_P)} d\mathbf{k}, \tag{56}$$

where the dielectric function takes into account the response of the whole system. The bare force due to the test particle alone is

$$\mathbf{F}(P \to 0) = -im \int \mathbf{k}\hat{u}(k) e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_P)} d\mathbf{k}. \tag{57}$$

If we subtract Eq. (57) from Eq. (56), we get the force created in $\mathbf{r}$ by the perturbation of the distribution function of the field particles caused by the introduction of the test particle. Evaluating this force at the position of the test particle, we obtain the friction that it experiences as a result of the polarization process

$$\mathbf{F}_{pol} = -im \int \mathbf{k}\hat{u}(k) \left[\frac{1}{\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})} - 1\right] d\mathbf{k}. \tag{58}$$

This can also be written

$$\mathbf{F}_{pol} = m \int \mathbf{k}\hat{u}(k) \mathrm{Im}\left[\frac{1}{\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})}\right] d\mathbf{k}. \tag{59}$$

Using the identity (B12) of Paper II, we finally obtain

$$F^{\mu}_{pol} = \pi(2\pi)^d m \int d\mathbf{v}_1 d\mathbf{k} \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} k^{\mu}k^{\nu}\delta(\mathbf{k}\cdot\mathbf{w})\frac{\partial f_1}{\partial v_1^{\nu}}. \tag{60}$$

If we neglect collective effects and take $|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2 = 1$, we recover Eq. (55) obtained in a different manner. Now, the friction force is due not only to the polarization but also to the variation of the diffusion coefficient with the velocity of the test particle $\mathbf{v}$. As a result, the complete expression of the friction force is

$$F^{\mu}_{friction} \equiv \left\langle \frac{\Delta v^{\mu}}{\Delta t} \right\rangle = F^{\mu}_{pol} + \frac{\partial D^{\mu\nu}}{\partial v^{\nu}}. \tag{61}$$

The second term is obtained when we take into account the influence of the fluctuations of the force in the trajectory of the test particle, i.e. when we go beyond the rectilinear trajectory approximation. As shown by Hubbard [47], this is necessary for the calculation of the friction while this is not necessary for the calculation of the diffusion coefficient. From Eqs. (42) and (60) we get

$$F^{\mu}_{friction} = \pi(2\pi)^d m \int d\mathbf{v}_1 d\mathbf{k} k^{\mu}k^{\nu}\hat{u}(k)^2 f_1 \left(\frac{\partial}{\partial v^{\nu}} - \frac{\partial}{\partial v_1^{\nu}}\right) \frac{\delta(\mathbf{k}\cdot\mathbf{w})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2}, \tag{62}$$

where we have used an integration by parts in Eq. (60). When we ignore collective effects, expressions (42) and (60) for the diffusion coefficient and the friction force can be obtained directly from the Hamiltonian equations, by making a systematic expansion of the trajectory of the particles in powers of $1/N$ in the limit $N \to +\infty$ as shown in Appendix A.

For a thermal bath, corresponding to the case where the field particles are at statistical equilibrium, the distribution function is the Maxwell-Boltzmann distribution

$$f(\mathbf{v}_1) = \left(\frac{\beta m}{2\pi}\right)^{d/2} \rho e^{-\beta m \frac{v_1^2}{2}}. \tag{63}$$

Inserting the identity

$$\frac{\partial f}{\partial \mathbf{v}_1} = -\beta m f_1 \mathbf{v}_1, \tag{64}$$

in Eq. (60), using the δ-function to replace $\mathbf{k} \cdot \mathbf{v}_1$ by $\mathbf{k} \cdot \mathbf{v}$, and comparing with Eq. (42), we find that

$$F^{\mu}_{pol} = -\beta m D^{\mu\nu} v^{\nu}. \tag{65}$$

This can be viewed as a generalized Einstein relation. We note that the diffusion coefficient and the friction coefficient depend on the velocity of the test particle. *We also note that the Einstein relation is valid for the friction force* $\mathbf{F}_{pol}$ *due to the polarization, not for the total friction force (62).* We do not have this subtlety for the ordinary Brownian motion where the diffusion coefficient is constant.

We now consider an arbitrary (steady) distribution of the bath. If we neglect collective effects and use Eqs. (40) and (55) we obtain after simple manipulations (see Eq. (16) in [16]):

$$\frac{\partial D^{\mu\nu}}{\partial v^{\nu}} = F^{\mu}_{pol}. \tag{66}$$

Therefore,

$$\mathbf{F}_{friction} = 2\mathbf{F}_{pol}. \tag{67}$$

We note that the friction force calculated by Kandrup [46] corresponds to the polarization part $\mathbf{F}_{pol}$ while Chandrasekhar [48] computes the full friction $\mathbf{F}_{friction}$. This explains why there is a factor 1/2 between their results for equal mass particles (see [46], pp. 446).

Finally, for 1D systems, we have the simple result

$$F_{pol} = 4\pi^2 m f'(v) \int_0^{+\infty} \frac{k\hat{u}(k)^2}{|\epsilon(k, kv)|^2} dk. \tag{68}$$

This expression is valid for an arbitrary (steady) distribution of the bath and it takes into account collective effects. Comparing Eq. (68) with Eq. (43), we find that the friction force is related to the diffusion coefficient by the relation

$$F_{pol} = D(v) \frac{d \ln f}{dv}. \tag{69}$$

This can be viewed as a generalization of the Einstein relation for an out-of-equilibrium distribution of the bath.

3.3 The Fokker-Planck equation

Assuming that the system is spatially homogeneous, the probability density $P(\mathbf{v}, t)$ of finding the test particle with the velocity $\mathbf{v}$ at time t is governed by a Fokker-Planck equation of the form

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial v^{\mu} \partial v^{\nu}} \left(P \frac{\langle \Delta v^{\mu} \Delta v^{\nu} \rangle}{\Delta t} \right) - \frac{\partial}{\partial v^{\mu}} \left(P \frac{\langle \Delta v^{\mu} \rangle}{\Delta t} \right). \tag{70}$$

This Fokker-Planck approach assumes that the stochastic process is markovian (see Sec. 4 for generalizations). It also assumes that the higher order moments of the increment of velocity Δv play a negligible role. This is indeed the case in the $N \to +\infty$ limit that we consider since they

are of order $O(N^{-2})$ or smaller. At order $O(N^{-1})$, we have found that the second (diffusion) and first (friction) moments of the velocity increment of the test particle are given by

$$\frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{2\Delta t} = D^{\mu\nu}, \qquad \frac{\langle \Delta v^\mu \rangle}{\Delta t} = \frac{\partial D^{\mu\nu}}{\partial v^\nu} + \eta^\mu, \tag{71}$$

with

$$D^{\mu\nu} = \pi(2\pi)^d m \int k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \delta(\mathbf{k}\cdot\mathbf{w}) f(\mathbf{v}_1) d\mathbf{k} d\mathbf{v}_1, \tag{72}$$

$$\eta^\mu \equiv F^\mu_{pol} = \pi(2\pi)^d m \int \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} k^\mu k^\nu \delta(\mathbf{k}\cdot\mathbf{w}) \frac{\partial f_1}{\partial v_1^\nu} d\mathbf{v}_1 d\mathbf{k}. \tag{73}$$

Note that we have changed the sign of η^μ with respect to Paper II. The Fokker-Planck equation (70) can be written in the alternative form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \left(D^{\mu\nu} \frac{\partial P}{\partial v^\nu} - P\eta^\mu \right). \tag{74}$$

The two expressions (70) and (74) have their own interest. The expression (70) where the diffusion coefficient is placed after the second derivative $\partial^2(DP)$ involves the total friction force $F^\mu_{friction} = \langle \Delta v^\mu \rangle / \Delta t$ and the expression (74) where the diffusion coefficient is placed between the derivatives $\partial D \partial P$ isolates the part of the friction $\eta^\mu = F^\mu_{pol}$ due to the polarization. This alternative form (74) has therefore a clear physical interpretation. Inserting the expressions (72) and (73) of the diffusion coefficient and friction term in Eq. (74), we obtain

$$\frac{\partial P}{\partial t} = \pi(2\pi)^d m \frac{\partial}{\partial v^\mu} \int d\mathbf{v}_1 d\mathbf{k} k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}_1)] \left(\frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v_1^\nu} \right) f(\mathbf{v}_1) P(\mathbf{v}, t). \tag{75}$$

For a thermal bath, using Eqs. (65), the Fokker-Planck equation (74) can be written

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \left[D^{\mu\nu} \left(\frac{\partial P}{\partial v^\nu} + \beta m P v^\nu \right) \right], \tag{76}$$

where $D^{\mu\nu}(v)$ is given by Eq. (72). Since the r.h.s. of Eq. (76) is of order $O(1/N)$, the distribution of the test particle $P(\mathbf{v}, t)$ relaxes to the Maxwellian distribution on a typical timescale Nt_D (see [16] for more details). In one dimension, the bath $f(v)$ can be any stable stationary solution of the Vlasov equation. Using Eq. (69), the Fokker-Planck equation (74) can be written

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left[D \left(\frac{\partial P}{\partial v} - P \frac{d \ln f}{dv} \right) \right], \tag{77}$$

where $D(v)$ is given by Eq. (43). The distribution of the test particle $P(v, t)$ relaxes to the distribution of the bath $f(v)$ on a typical timescale Nt_D [16].

In Paper II, we have obtained the Fokker-Planck equation (75) from the Lenard-Balescu equation (II-49) by replacing $f(\mathbf{v}, t)$ by the distribution of the test particle $P(\mathbf{v}, t)$ and $f(\mathbf{v}_1, t)$ by the static distribution of the bath $f(\mathbf{v}_1)$. This procedure transforms an integrodifferential equation (II-49) in a differential equation (75). The expressions (71)-(73) of the diffusion and friction were then obtained by identifying Eq. (75) with the Fokker-Planck equation (70). In the present paper, we have proceeded the other way round by first determining the moments (71)-(73), then inserting them in the Fokker-Planck equation (70). Note that Hubbard [47] derived the expressions (71)-(73) of the diffusion coefficient and friction force but did not make the calculations explicitly until the end. In particular, he did not explicitly wrote down the kinetic equation (75) that is related to the Lenard-Balescu equation (II-49) discovered independently at the same period [49, 50].

3.4 The Fokker-Planck equation at $T = 0$

In Paper II and in [16], we have given various expressions of the Fokker-Planck equation (75) for different potentials of interaction and different dimensions of space. However, we have not explicitly considered the case $T = 0$ which presents interesting features. At $T = 0$, the Maxwell-Boltzmann distribution (63) reduces to $f(\mathbf{v}_1) = \rho\delta(\mathbf{v}_1)$. Substituting this expression in Eq. (72), we find that the diffusion coefficient becomes

$$D^{\mu\nu} = \pi(2\pi)^d \rho m \int k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k},0)|^2}\delta(\mathbf{k}\cdot\mathbf{v})d\mathbf{k}, \tag{78}$$

with $\epsilon(\mathbf{k},0) = 1 + (2\pi)^d\hat{u}(k)\beta\rho m$ according to Eq. (II-13). Note that, in this section, we consider the case of repulsive potentials with $\hat{u}(k) > 0$ so that the homogeneous phase is stable even at $T = 0$ (see Paper I). We now observe that the integral in Eq. (78) is similar to the one in Eq. (II-41). Therefore, it can be written

$$D^{\mu\nu} = \frac{K_d}{v}\left(\delta^{\mu\nu} - \frac{v^\mu v^\nu}{v^2}\right), \tag{79}$$

where

$$K_d = \lambda_d \rho m \int_0^{+\infty} k^d \left[\frac{\hat{u}(k)}{1 + (2\pi)^d\hat{u}(k)\beta\rho m}\right]^2 dk, \tag{80}$$

with $\lambda_3 = 8\pi^5$ and $\lambda_2 = 8\pi^3$. For the Coulombian potential, we have $(2\pi)^3\hat{u}(k)\beta\rho m = k_D^2/k^2$ where k_D is the Debye wavenumber (see Paper I). Therefore, the collective effects encapsulated in the dielectric function in the denominator of Eq. (80) regularise the integral for $k \to 0$ (this is a particular case of the Lenard-Balescu equation). On the other hand, noting that $D^{\mu\nu}v^\nu = 0$ according to Eq. (79), we find that the friction force (65) vanishes. Therefore, at $T = 0$, the Fokker-Planck equation (75) can be written

$$\frac{\partial P}{\partial t} = K_d \frac{\partial}{\partial v^\mu}\left(\frac{\delta^{\mu\nu}v^2 - v^\mu v^\nu}{v^3}\frac{\partial P}{\partial v^\nu}\right). \tag{81}$$

This equation admits an infinity of stationary solutions. Indeed, since $D^{\mu\nu}v^\nu = 0$, any distribution $P = P(v)$ depending only on the modulus $v = |\mathbf{v}|$ of the velocity is a stationary solution of Eq. (81). Therefore, at $T = 0$, the test particle does not necessarily relax to the distribution of the bath $f(\mathbf{v}) = \rho\delta(\mathbf{v})$. On the other hand, for one dimensional systems, Eq. (81) reduces to $\partial P/\partial t = 0$ so that the distribution of the test particle does not evolve in time.

3.5 More general kinetic equations

It is instructive to compare the Fokker-Planck equation (75) with the more general equation obtained from the projection operator formalism [23]. When collective effects are ignored, this equation can be written

$$\frac{\partial P}{\partial t} + \mathbf{v}\frac{\partial P}{\partial \mathbf{r}} + \langle \mathbf{F}\rangle \frac{\partial P}{\partial \mathbf{v}} = \frac{\partial}{\partial v^\mu}\int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 F^\mu(1\to 0)G(t,t-\tau)$$
$$\times\left\{\mathcal{F}^\nu(1\to 0)\frac{\partial}{\partial v^\nu} + \mathcal{F}^\nu(0\to 1)\frac{\partial}{\partial v_1^\nu}\right\}P(\mathbf{r},\mathbf{v},t-\tau)\frac{f}{m}(\mathbf{r}_1,\mathbf{v}_1). \tag{82}$$

It can be obtained from Eq. (13) by replacing $f(\mathbf{v},t)$ by $P(\mathbf{v},t)$ and $f(\mathbf{v}_1,t)$ by $f(\mathbf{v}_1)$. This is a sort of generalized "Fokker-Planck" equation involving a term of "diffusion" and a term of

"friction". However, strictly speaking, Eq. (82) is not a Fokker-Planck equation because it is non-Markovian. We also note that the "diffusion" term appears as a complicated time integral of the force correlation function involving $P(\mathbf{r},\mathbf{v},t-\tau)$. This can be seen as a generalization of the Kubo formula (35). Similarly the "friction" force is a generalization of the expression obtained in Eq. (49) with a more complicated time integral. If we consider a thermal bath where the distribution of the field particles is the Boltzmann distribution, we get

$$\frac{\partial P}{\partial t}+\mathbf{v}\frac{\partial P}{\partial \mathbf{r}}+\langle \mathbf{F}\rangle\frac{\partial P}{\partial \mathbf{v}}=\frac{\partial}{\partial v^{\mu}}\int_0^t d\tau\int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0)G(t,t-\tau)$$
$$\times\left\{\mathcal{F}(1\to 0)\cdot\frac{\partial}{\partial \mathbf{v}}-\beta m\mathcal{F}(0\to 1)\cdot\mathbf{v}_1\right\}P(\mathbf{r},\mathbf{v},t-\tau)\frac{f}{m}(\mathbf{r}_1,\mathbf{v}_1). \quad (83)$$

If we come back to Eq. (82), make a Markovian approximation and extend the time integration to infinity, we get

$$\frac{\partial P}{\partial t}+\mathbf{v}\frac{\partial P}{\partial \mathbf{r}}+\langle \mathbf{F}\rangle\frac{\partial P}{\partial \mathbf{v}}=\frac{\partial}{\partial v^{\mu}}\int_0^{+\infty} d\tau\int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0)G(t,t-\tau)$$
$$\times\left\{\mathcal{F}^{\nu}(1\to 0)\frac{\partial}{\partial v^{\nu}}+\mathcal{F}^{\nu}(0\to 1)\frac{\partial}{\partial v_1^{\nu}}\right\}P(\mathbf{r},\mathbf{v},t)\frac{f}{m}(\mathbf{r}_1,\mathbf{v}_1), \quad (84)$$

where we recall that the coordinates appearing after the Greenian must be viewed as explicit functions of time $\mathbf{r}_i(t-\tau)$ and $\mathbf{v}_i(t-\tau)$ (see Paper III for more details). For a spatially homogeneous system, Eq. (82) takes the simplest form

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v^{\mu}}\int_0^t d\tau\int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\left(\frac{\partial}{\partial v^{\nu}}-\frac{\partial}{\partial v_1^{\nu}}\right)P(\mathbf{v},t-\tau)\frac{f}{m}(\mathbf{v}_1), \quad (85)$$

where we have used $\mathbf{v}_i(t-\tau)=\mathbf{v}_i$ for a spatially homogeneous system. We shall come back to this non-markovian equation in Sec. 4. If we now make a Markovian approximation $P(\mathbf{v},t-\tau)\simeq P(\mathbf{v},t)$ and extend the time integral to infinity, we get

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v^{\mu}}\int_0^{+\infty} d\tau\int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\left(\frac{\partial}{\partial v^{\nu}}-\frac{\partial}{\partial v_1^{\nu}}\right)P(\mathbf{v},t)\frac{f}{m}(\mathbf{v}_1). \quad (86)$$

This is a Fokker-Planck equation which can be put in the form (74) with a diffusion coefficient

$$D^{\mu\nu}=\int_0^{+\infty} d\tau\int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\frac{f}{m}(\mathbf{v}_1), \quad (87)$$

and a friction force due to the polarization

$$\eta^{\mu}=-\frac{1}{m}\int_0^{+\infty} d\tau\int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)F^{\nu}(0\to 1,t-\tau)\frac{\partial f}{\partial v^{\nu}}(\mathbf{v}_1). \quad (88)$$

These expressions agree with Eqs. (36) and (54) obtained directly from the equations of motion. After integration on τ and $\mathbf{r}_1$, we recover the Fokker-Planck equation (75) with the expressions (72) and (73) of the diffusion coefficient and friction term (with $|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2=1$ since collective effects are neglected here).

4 The non-markovian equation

4.1 General results

In this section, we study in more detail the non-Markovian equation (85). If the field particles are at statistical equilibrium (thermal bath), using the identity (64), the non-markovian equation (85) takes the form

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v^{\mu}}\int_0^t d\tau\int d\mathbf{r}_1 d\mathbf{v}_1 F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\frac{f}{m}(\mathbf{v}_1)\left(\frac{\partial}{\partial v^{\nu}}+\beta m v_1^{\nu}\right)P(\mathbf{v},t-\tau). \tag{89}$$

It can be rewritten

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v^{\mu}}\int_0^t d\tau\left(C^{\mu\nu}(\tau)\frac{\partial}{\partial v^{\nu}}+\beta m W^{\mu}(\tau)\right)P(\mathbf{v},t-\tau), \tag{90}$$

where we have introduced the notations

$$C^{\mu\nu}(\tau)=\langle F^{\mu}(t)F^{\nu}(t-\tau)\rangle=N\langle F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\rangle, \tag{91}$$

$$W^{\mu}(\tau)=N\langle F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)v_1^{\nu}\rangle. \tag{92}$$

These quantities can be calculated by making the linear trajectory approximation. The first quantity has already been studied in Paper II. It represents the temporal correlation of the force acting on the test particle. It can be written

$$C^{\mu\nu}(\tau)=(2\pi)^d m\int k^{\mu}k^{\nu}\hat{u}(k)^2 e^{-i\mathbf{k}(\mathbf{v}-\mathbf{v}_1)\tau}f(\mathbf{v}_1)d\mathbf{v}_1 d\mathbf{k}. \tag{93}$$

Performing the integration on $\mathbf{v}_1$, we get

$$C^{\mu\nu}(\tau)=(2\pi)^{2d} m\int k^{\mu}k^{\nu}\hat{u}(k)^2 e^{-i\mathbf{k}\mathbf{v}\tau}\hat{f}(\mathbf{k}\tau)d\mathbf{k}, \tag{94}$$

where $\hat{f}$ is the Fourier transform of f. For a Maxwellian distribution of the field particles (thermal bath), we have

$$C^{\mu\nu}(\tau)=(2\pi)^d \rho m\int k^{\mu}k^{\nu}\hat{u}(k)^2 e^{-i\mathbf{k}\mathbf{v}\tau}e^{-k^2\tau^2/2\beta m}d\mathbf{k}. \tag{95}$$

On the other hand, the function $\mathbf{W}(\tau)$ is given by

$$W^{\mu}(\tau)=m(2\pi)^d\int(\mathbf{k}\cdot\mathbf{v}_1)k^{\mu}\hat{u}(k)^2 e^{-i\mathbf{k}(\mathbf{v}-\mathbf{v}_1)\tau}f(\mathbf{v}_1)d\mathbf{v}_1 d\mathbf{k}. \tag{96}$$

Performing the integration on $\mathbf{v}_1$, we get

$$W^{\mu}(\tau)=-im(2\pi)^{2d}\int d\mathbf{k}k^{\mu}\hat{u}(k)^2 e^{-i\mathbf{k}\mathbf{v}\tau}\frac{\partial}{\partial\tau}\hat{f}(\mathbf{k}\tau). \tag{97}$$

For a Maxwellian distribution of the field particles, we obtain

$$W^{\mu}(\tau)=-i\rho m(2\pi)^d\int d\mathbf{k}k^{\mu}\hat{u}(k)^2 e^{-i\mathbf{k}\mathbf{v}\tau}\frac{\partial}{\partial\tau}e^{-k^2\tau^2/2\beta m}, \tag{98}$$

so that, finally,

$$W^{\mu}(\tau)=i(2\pi)^d\rho\frac{\tau}{\beta}\int k^{\mu}\hat{u}(k)^2 k^2 e^{-i\mathbf{k}\mathbf{v}\tau}e^{-k^2\tau^2/2\beta m}d\mathbf{k}. \tag{99}$$

Let us now apply these general results to some specific systems.

4.2 Self-gravitating systems

For the gravitational interaction, we can easily perform the integrations in Eqs. (95) and (99) by introducing a spherical system of coordinates with the z axis in the direction of $\mathbf{v}$. The correlation function $C^{\mu\nu}(\tau)$ is given by Eqs. (II-94), (II-95) and (II-96). On the other hand, after some calculations, we find that

$$\mathbf{W}(\tau) = \frac{4\pi\rho m G^2}{v\tau} G(x)\mathbf{v}, \tag{100}$$

where $\mathbf{x} = (\beta m/2)^{1/2}\mathbf{v}$ and $G(x)$ is the function defined by Eq. (II-75). Comparing this expression with Eq. (II-95), we find that

$$\mathbf{W}(\tau) = C_{\parallel}(v,\tau)\mathbf{v} = C^{\mu\nu}(v,\tau)v^{\nu}. \tag{101}$$

Therefore, for the gravitational interaction, we have the equality

$$\langle F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)v_1^{\nu}\rangle = \langle F^{\mu}(1\to 0,t)F^{\nu}(1\to 0,t-\tau)\rangle v^{\nu}. \tag{102}$$

We stress, however, that this equality is not true for any potential. Using the relation (101), we can rewrite the non-Markovian equation (90) in the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^{\mu}}\int_0^t d\tau C^{\mu\nu}(\tau)\left(\frac{\partial}{\partial v^{\nu}} + \beta m v^{\nu}\right)P(\mathbf{v},t-\tau). \tag{103}$$

For a spherically symmetric system, the distribution $P(\mathbf{v},t)$ depends only on the modulus $|\mathbf{v}| = v$ of the velocity and we obtain

$$\frac{\partial P}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\int_0^t d\tau C_{\parallel}(\tau,v)\left(\frac{\partial}{\partial v} + \beta m v\right)P(v,t-\tau)\right], \tag{104}$$

where (see Paper II):

$$C_{\parallel}(\tau,v) = \frac{4\pi\rho m G^2}{v\tau}G(x). \tag{105}$$

If we make a markovian approximation $P(v,t-\tau) \simeq P(v,t)$ and extend the time integration to $+\infty$, we recover the Kramers-Chandrasekhar equation [48]:

$$\frac{\partial P}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2 D_{\parallel}(v)\left(\frac{\partial P}{\partial v} + \beta m P v\right)\right], \tag{106}$$

with

$$D_{\parallel}(v) = \int_0^{+\infty} C_{\parallel}(\tau,v)d\tau = \frac{4\pi\rho m G^2}{v}G(x)\int_0^{+\infty}\frac{d\tau}{\tau}. \tag{107}$$

This expression exhibits the well-known logarithmic divergence of the diffusion coefficient which appears here in the time integration (see a discussion of this issue in Paper II). The divergence for $t \to 0$ is related to the linear trajectory approximation and could be cured by a more accurate treatment of binary collisions. The divergence for $t \to +\infty$ is more serious. One usually introduces a cut-off but this procedure is relatively *ad hoc*. Alternatively, one could consider the non-Markovian equation (104)-(105) which is well-posed for any time t.

4.3 The HMF model

As discussed previously, non-markovian effects can be important for self-gravitating systems because the temporal correlation function of the force decreases algebraically, like t^{-1}. For neutral plasmas, the situation is different because of Debye shielding. In that case, collective effects cannot be ignored in the computation of the force auto-correlation function and they are taken into account through the dielectric function in Eq. (II-98). Then, the temporal correlation function is given by Eqs. (II-100), (II-109), (II-110) and (II-21) of Paper II, and it decreases exponentially rapidly. In that case, the Markovian approximation is valid. Collective effects are also important for the HMF model [51, 52, 3]. When collective effects are ignored, it is found that the temporal decay of the correlation function of the force is gaussian, see Eq. (II-113). By contrast, when collective effects are properly accounted for, it is found that the correlation function decreases exponentially rapidly, see Eq. (II-111). Furthermore, the decay rate tends to zero for $T \to T_c$ implying a slow decay of the correlations. This may unveil a failure of the markovian approximation close to the critical temperature. For that reason, it may be useful to derive non-markovian kinetic equations which take into account collective effects.

Collective effects can be taken into account in the non-markovian equation (90) by making the substitution

$$\hat{u}(k)^2 \to \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1)|^2}, \tag{108}$$

in the expressions (93) and (96). The correlation function of the force is now given by

$$C^{\mu\nu}(\tau) = (2\pi)^d m \int k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1)|^2} e^{-i\mathbf{k}(\mathbf{v}-\mathbf{v}_1)\tau} f(\mathbf{v}_1) d\mathbf{v}_1 d\mathbf{k}. \tag{109}$$

This can be written

$$C^{\mu\nu}(\tau) = (2\pi)^d m \int k^\mu k^\nu \hat{u}(k)^2 e^{-i\mathbf{k}\mathbf{v}\tau} Q(\mathbf{k}, \tau) d\mathbf{k}, \tag{110}$$

where the function $Q(\mathbf{k}, \tau)$ is defined by Eq. (II-99). On the other hand, the function $\mathbf{W}(\tau)$ is given by

$$W^\mu(\tau) = m(2\pi)^d \int (\mathbf{k}\cdot\mathbf{v}_1) k^\mu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1)|^2} e^{-i\mathbf{k}(\mathbf{v}-\mathbf{v}_1)\tau} f(\mathbf{v}_1) d\mathbf{v}_1 d\mathbf{k}. \tag{111}$$

This can be rewritten

$$W^\mu(\tau) = -im(2\pi)^d \int d\mathbf{k} k^\mu \hat{u}(k)^2 e^{-i\mathbf{k}\mathbf{v}\tau} \frac{\partial}{\partial\tau} Q(\mathbf{k}, \tau). \tag{112}$$

For a Maxwellian distribution of the field particles, the large time asymptotics of $Q(\mathbf{k}, \tau)$ is given by (II-109). Using Eqs. (110) and (112), we can then obtain the large time asymptotic of $C^{\mu\nu}(\tau)$ and $W^\mu(\tau)$ for $\tau \to +\infty$.

Let us now specifically consider the HMF model where the potential of interaction is truncated to one Fourier mode. For this system, using Eqs. (110) and (112), the non-markovian equation (90) can be written

$$\frac{\partial P}{\partial t} = \frac{k^2}{4\pi} \frac{\partial}{\partial v} \int_0^t d\tau \left[Q(\tau) \cos(v\tau) \frac{\partial}{\partial v} - \beta Q'(\tau) \sin(v\tau) \right] P(v, t-\tau), \tag{113}$$

where $Q(\tau)$ behaves like

$$Q(\tau) \sim \rho \left(\frac{2}{\beta}\right)^{1/2} \frac{1}{\gamma|F'(\gamma\sqrt{\beta/2})|} e^{-\gamma\tau}, \tag{114}$$

for $\tau \to +\infty$. The damping rate γ and the function $F(x)$ are defined in Paper II. As discussed above, the exponential relaxation time $\gamma^{-1}(T)$ diverges for $T \to T_c$ so that the Markovian approximation may not be correct close to the critical point. This may be an interesting situation to analyze in deeper detail with the non-markovian equation (113).

If we neglect collective effects, we find that

$$Q(\mathbf{k}, \tau) = (2\pi)^d \hat{f}(\mathbf{k}t) = \rho e^{-k^2\tau^2/2\beta m}, \tag{115}$$

where the second equality is valid for a Maxwellian distribution of the field particles. For the HMF model, we obtain

$$Q(\tau) = \rho e^{-\tau^2/2\beta}. \tag{116}$$

This yields a Gaussian decay of the correlations instead of an exponential decay in Eq. (114) when collective effects are accounted for [52]. With the expression (114) for $Q(\tau)$, the non-markovian equation (90) becomes

$$\frac{\partial P}{\partial t} = \frac{\rho k^2}{4\pi}\frac{\partial}{\partial v}\int_0^t d\tau e^{-\tau^2/(2\beta)}\left[\cos(v\tau)\frac{\partial}{\partial v} + \tau\sin(v\tau)\right] P(v, t-\tau). \tag{117}$$

We note that for the HMF model, the equality (102) is not satisfied.

The previous equations assume that the distribution of the bath is maxwellian. More generally, if we come back to the non-Markovian equation Eq. (85) and perform the integration on $\mathbf{r}_1$, we obtain

$$\frac{\partial P}{\partial t} = m(2\pi)^d \frac{\partial}{\partial v^\mu}\int_0^t d\tau \int d\mathbf{v}_1 d\mathbf{k} k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1)|^2}\cos(\mathbf{k}\cdot\mathbf{w}\tau)\left(\frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v_1^\nu}\right) P(\mathbf{v}, t-\tau) f(\mathbf{v}_1). \tag{118}$$

For the HMF model, this equation reduces to

$$\frac{\partial P}{\partial t} = \frac{k^2}{4\pi}\frac{\partial}{\partial v}\int_0^t d\tau \int dv_1 \frac{\cos(w\tau)}{|\epsilon(1, v_1)|^2}\left(\frac{\partial}{\partial v} - \frac{d}{dv_1}\right) P(v, t-\tau) f(v_1). \tag{119}$$

This equation is valid for any steady distribution of the field particles, not only for the statistical equilibrium state (thermal bath).

5 Conclusion

In this paper, starting from the Klimontovich equation and using a quasilinear theory, we have developed a kinetic theory for systems with weak long-range interactions. We have obtained general equations that take into account spatial inhomogeneity and memory effects. These peculiarities are specific to systems with unshielded long-range interactions. However, in order to obtain closed kinetic equations we have been obliged to neglect some collective effects. These collective effects can be taken into account for spatially homogeneous systems with short memory time (with respect to the slow collisional relaxation time). In that case, we recover

the Lenard-Balescu equation of plasma physics with slight modifications (see Paper II and footnote 2). It would be valuable to develop a formalism that takes into account both spatial inhomogeneity and collective effects. This has been partly done in Paper III, where we have obtained two coupled Eqs. (III-6)-(III-7) that are exact at order $O(1/N)$. However, it seems difficult to go any further without either (i) considering homogeneous systems or (ii) neglecting collective effects. In fact, due to the huge timescale separation between the dynamical time t_D and the relaxation time $\geq Nt_D$, it could be of interest to develop a kinetic theory in angle-action variables as considered in [53], where an orbit-averaged-Fokker-Planck equation has been derived for one-dimensional systems with weak long-range interactions (note that the formalism developed in the present paper could be used to better justify the kinetic equation obtained in [53]). This could be a future direction of investigation. Another direction of research would be to investigate in deeper detail the non-markovian kinetic equations derived in this paper. This will be considered in future works.

A First and second moments of the velocity increment

In this Appendix, we calculate the first and second moments $\langle \Delta v^{\mu} \rangle$ and $\langle \Delta v^{\mu} \Delta v^{\nu} \rangle$ of the increment of velocity of the test particle directly from the Hamiltonian equations of motion (I-1). We follow a procedure similar to that used by Valageas [54] in a different context. Since the calculations are similar, we shall only give the main steps of the derivation. For simplicity, we assume that all the particles have the same mass m. In order to separate the mean field dynamics from the discrete effects giving rise to the diffusion and to the friction of the test particle, we write the Hamiltonian (I-1) as

$$H = m(H_0 + H_I), \tag{120}$$

where we defined the mean field Hamiltonian H_0 by

$$H_0 = \frac{1}{2} \sum_i v_i^2 + \sum_i \Phi_0(\mathbf{r}_i), \tag{121}$$

and the interaction Hamiltonian H_I by

$$H_I = e^{\omega t} \left[m \sum_{i<j} u(\mathbf{r}_i - \mathbf{r}_j) - \sum_i \Phi_0(\mathbf{r}_i) \right]. \tag{122}$$

In Eqs. (121)-(122) the mean field potential is given by

$$\Phi_0(\mathbf{r}) = \int \rho(\mathbf{r}') u(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}', \tag{123}$$

where $\rho(\mathbf{r}')$ is the mean field equilibrium spatial density of the particles. The factor $e^{\omega t}$ has been added for the computation of perturbative eigenmodes and we shall ultimately let $\omega \to 0^+$. Thus H_0 describes the mean field dynamics whereas H_I describes the discrete effects which vanish in the limit $N \to +\infty$. Therefore, we consider H_I as a perturbation of H_0 and we apply a perturbative analysis in powers of $1/N$. We assume that the system is spatially homogeneous so that $\Phi_0 = 0$. The interaction Hamiltonian H_I can be rewritten

$$H_I = e^{\omega t} m \sum_{i<j} \int \hat{u}(\mathbf{k}) e^{i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)} d\mathbf{k}. \tag{124}$$

On the other hand, the equations of motion read

$$\frac{d\mathbf{v}_i}{dt} = -\frac{1}{m}\frac{\partial H}{\partial \mathbf{r}_i} = -\frac{\partial}{\partial \mathbf{r}_i}(H_0 + H_I), \qquad \frac{d\mathbf{r}_i}{dt} = \frac{1}{m}\frac{\partial H}{\partial \mathbf{v}_i} = \mathbf{v}_i. \tag{125}$$

We write the trajectories $\{\mathbf{r}(t), \mathbf{v}(t)\}$ as the perturbative expansions $\mathbf{r} = \mathbf{r}^{(0)} + \mathbf{r}^{(1)} + \mathbf{r}^{(2)} + ...$ where $\mathbf{r}^{(k)}$ is formally of order k over H_I. At zeroth-order, we simply have

$$\frac{d\mathbf{v}_i^{(0)}}{dt} = \mathbf{0}, \qquad \frac{d\mathbf{r}_i^{(0)}}{dt} = \mathbf{v}_i^{(0)}, \tag{126}$$

which yields the rectilinear orbits

$$\mathbf{v}_i^{(0)}(t') = \mathrm{Cte} = \mathbf{v}_i, \qquad \mathbf{r}_i^{(0)}(t') = \mathbf{v}_i(t'-t) + \mathbf{r}_i, \tag{127}$$

where, in the following, $\mathbf{r}_i$ and $\mathbf{v}_i$ denote the position and the velocity of the particle i at time t. At first order, we obtain

$$\frac{d\mathbf{v}_i^{(1)}}{dt} = -\frac{\partial H_I}{\partial \mathbf{r}_i}, \qquad \frac{d\mathbf{r}_i^{(1)}}{dt} = \mathbf{v}_i^{(1)}, \tag{128}$$

where we can substitute the zeroth-order orbits in the r.h.s. of these expressions. This yields

$$\frac{d\mathbf{v}_i^{(1)}}{dt} = -\frac{\partial}{\partial \mathbf{r}_i} e^{\omega t} m \sum_{j<j'} \int \hat{u}(k) e^{i\mathbf{k}(\mathbf{r}_j^{(0)} - \mathbf{r}_{j'}^{(0)})} d\mathbf{k}. \tag{129}$$

Using Eq. (127) and integrating on time, we obtain

$$\mathbf{v}_i^{(1)} = -\frac{\partial}{\partial \mathbf{r}_i} \int_{-\infty}^{t} dt' e^{\omega t'} m \sum_{j<j'} \int \hat{u}(k) e^{i\mathbf{k}(\mathbf{v}_j - \mathbf{v}_{j'})(t'-t)} e^{i\mathbf{k}(\mathbf{r}_j - \mathbf{r}_{j'})} d\mathbf{k}. \tag{130}$$

Therefore,

$$\mathbf{v}_i^{(1)} = -\frac{\partial}{\partial \mathbf{r}_i} e^{\omega t} m \sum_{j<j'} \int \hat{u}(k) \frac{e^{i\mathbf{k}(\mathbf{r}_j - \mathbf{r}_{j'})}}{\omega + i\mathbf{k}(\mathbf{v}_j - \mathbf{v}_{j'})} d\mathbf{k}. \tag{131}$$

Substituting this expression in Eq. (128)-b, we get

$$\frac{d\mathbf{r}_i^{(1)}}{dt} = -e^{\omega t} m \sum_{j\neq i} \int \hat{u}(k) \frac{i\mathbf{k} e^{i\mathbf{k}(\mathbf{r}_i^{(0)} - \mathbf{r}_j^{(0)})}}{\omega + i\mathbf{k}(\mathbf{v}_i^{(0)} - \mathbf{v}_j^{(0)})} d\mathbf{k}. \tag{132}$$

Using Eq. (127) and integrating on time again, we obtain

$$\mathbf{r}_i^{(1)} = -e^{\omega t} m \sum_{j\neq i} \int \hat{u}(k) \frac{i\mathbf{k} e^{i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)}}{[\omega + i\mathbf{k}(\mathbf{v}_i - \mathbf{v}_j)]^2} d\mathbf{k}. \tag{133}$$

This can be rewritten

$$\mathbf{r}_i^{(1)} = \frac{\partial}{\partial \mathbf{v}_i} e^{\omega t} m \sum_{j<j'} \int \hat{u}(k) \frac{e^{i\mathbf{k}(\mathbf{r}_j - \mathbf{r}_{j'})}}{\omega + i\mathbf{k}(\mathbf{v}_j - \mathbf{v}_{j'})} d\mathbf{k}. \tag{134}$$

Therefore, at first order, we have

$$\mathbf{v}_i^{(1)} = -\frac{\partial \chi}{\partial \mathbf{r}_i}, \qquad \mathbf{r}_i^{(1)} = \frac{\partial \chi}{\partial \mathbf{v}_i}, \tag{135}$$

with

$$\chi = e^{\omega t} m \sum_{j<j'} \int \hat{u}(k) \frac{e^{i\mathbf{k}(\mathbf{r}_j - \mathbf{r}_{j'})}}{\omega + i\mathbf{k}(\mathbf{v}_j - \mathbf{v}_{j'})} d\mathbf{k}. \tag{136}$$

At second order, we have

$$\frac{dv_i^{\mu(2)}}{dt} = -\sum_j \frac{\partial^2 H_I}{\partial r_i^\mu \partial r_j^\nu} r_j^{\nu(1)}. \tag{137}$$

The average acceleration of the test particle is

$$\langle \dot{v}^{\mu(2)} \rangle = -\left\langle \frac{\partial^2 H_I}{\partial r^\mu \partial r^\nu} r^{\nu(1)} \right\rangle - N \left\langle \frac{\partial^2 H_I}{\partial r^\mu \partial r_1^\nu} r_1^{\nu(1)} \right\rangle. \tag{138}$$

Using Eq. (134), the relations

$$\frac{\partial^2 H_I}{\partial r^\mu \partial r^\nu} = -e^{\omega t} m \sum_{j\neq 0} \int \hat{u}(\mathbf{k}) k^\mu k^\nu e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_j)} d\mathbf{k}, \tag{139}$$

$$\frac{\partial^2 H_I}{\partial r^\mu \partial r_1^\nu} = e^{\omega t} m \int \hat{u}(\mathbf{k}) k^\mu k^\nu e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_1)} d\mathbf{k}, \tag{140}$$

and performing the averages in Eq. (138) with respect to the distribution function $f(\mathbf{v})$, we obtain after some calculations

$$\begin{aligned}\langle \dot{v}^{\mu(2)} \rangle = (2\pi)^d m e^{2\omega t} \frac{\partial}{\partial v^\nu} \int d\mathbf{v}_1 d\mathbf{k} \hat{u}(k)^2 k^\mu k^\nu \frac{\omega}{\omega^2 + (\mathbf{k}\cdot\mathbf{w})^2} f(\mathbf{v}_1) \\ -(2\pi)^d m e^{2\omega t} \int d\mathbf{v}_1 d\mathbf{k} f(\mathbf{v}_1) \frac{\partial}{\partial v^\nu} \hat{u}(k)^2 k^\mu k^\nu \frac{\omega}{\omega^2 + (\mathbf{k}\cdot\mathbf{w})^2}.\end{aligned} \tag{141}$$

We introduce the velocity increment $\Delta v^\mu = v^\mu(t+\Delta t) - v^\mu(t)$. Noting that Eq. (141) represents the variation of the velocity increment at order $1/N$, taking the limit $\omega \to 0^+$ and using $\lim_{\omega\to 0}\omega/(\omega^2 + x^2) = \pi\delta(x)$, we find that

$$\left\langle \frac{\Delta v^\mu}{\Delta t} \right\rangle = \pi (2\pi)^d m \int d\mathbf{v}_1 d\mathbf{k} f(\mathbf{v}_1) \hat{u}(k)^2 k^\mu k^\nu \left(\frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v_1^\nu} \right) \delta(\mathbf{k}\cdot\mathbf{w}). \tag{142}$$

On the other hand, the diffusion tensor $\langle \Delta v^\mu \Delta v^\nu \rangle$ at order $1/N$ is equal to $\langle \Delta v^{\mu(1)} \Delta v^{\nu(1)} \rangle$. According to Eq. (131), we have

$$v^{\mu(1)} = -i e^{\omega t} m \sum_{j\neq 0} \int \hat{u}(k) k^\mu \frac{e^{i\mathbf{k}(\mathbf{r}^{(0)} - \mathbf{r}_j^{(0)})}}{\omega + i\mathbf{k}(\mathbf{v}^{(0)} - \mathbf{v}_j^{(0)})} d\mathbf{k}. \tag{143}$$

Therefore, using Eq. (127), we obtain

$$\Delta v^{\mu(1)} = -ie^{\omega(t+\Delta t)} m \sum_{j\neq 0} \int d\mathbf{k}\hat{u}(k)k^\mu \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_j)+i\mathbf{k}(\mathbf{v}-\mathbf{v}_j)\Delta t}}{\omega + i\mathbf{k}(\mathbf{v}-\mathbf{v}_j)}$$
$$+ie^{\omega t} m \sum_{j\neq 0} \int d\mathbf{k}\hat{u}(k)k^\mu \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_j)}}{\omega + i\mathbf{k}(\mathbf{v}-\mathbf{v}_j)}. \quad (144)$$

Taking the average with respect to the distribution function $f(\mathbf{v})$, we obtain after some calculations

$$\left\langle \frac{\Delta v^\mu \Delta v^\nu}{\Delta t} \right\rangle = (2\pi)^d m e^{2\omega t} \int d\mathbf{k} d\mathbf{v}_1 f(\mathbf{v}_1) k^\mu k^\nu \frac{\hat{u}(k)^2}{\omega^2 + (\mathbf{k}\cdot\mathbf{w})^2}$$
$$\times \frac{1}{\Delta t}\left[1 + e^{2\omega\Delta t} - 2e^{\omega\Delta t}\cos(\mathbf{k}\cdot\mathbf{w}\Delta t)\right]. \quad (145)$$

The limit $\omega \to 0^+$ now gives

$$\left\langle \frac{\Delta v^\mu \Delta v^\nu}{\Delta t} \right\rangle = 2(2\pi)^d m \int d\mathbf{k} d\mathbf{v}_1 f(\mathbf{v}_1) k^\mu k^\nu \frac{\hat{u}(k)^2}{\Delta t(\mathbf{k}\cdot\mathbf{w})^2}\left[1 - \cos(\mathbf{k}\cdot\mathbf{w}\Delta t)\right]. \quad (146)$$

Finally, taking $\Delta t \to +\infty$ and using $\lim_{t\to+\infty}(1-\cos tx)/tx^2 = \pi\delta(x)$, we obtain

$$\left\langle \frac{\Delta v^\mu \Delta v^\nu}{\Delta t} \right\rangle = 2\pi(2\pi)^d m \int d\mathbf{k} d\mathbf{v}_1 f(\mathbf{v}_1) k^\mu k^\nu \hat{u}(k)^2 \delta(\mathbf{k}\cdot\mathbf{w}). \quad (147)$$

This relation can also be obtained from the Kubo formula $\int_0^{+\infty} \dot{v}^{(1)\mu}(t)\dot{v}^{(1)\nu}(t+\Delta t)d(\Delta t)$ using Eqs. (127) and (129). Equations (142) and (147) return the terms of diffusion and friction obtained in Sec. 3 when collective effects are neglected.

B Collective effects

When collective effects are taken into account, the force created by a particle on the others is modified by the influence of a "polarization cloud". This effect can be calculated precisely in the case of a spatially homogeneous medium. In that case, the linearized Klimontovich equations are

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f}{\partial \mathbf{r}} - \nabla\delta\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \quad (148)$$

$$\delta\Phi(\mathbf{r},t) = \int u(\mathbf{r}-\mathbf{r}')\left[\delta\rho(\mathbf{r}',t) + m\delta(\mathbf{r}'-\mathbf{r}_1-\mathbf{v}_1 t)\right]d\mathbf{r}', \quad (149)$$

where $(\mathbf{r}_1, \mathbf{v}_1)$ represent the position and the velocity of the test particle (here denoted 1) at time $t = 0$, and we have made the linear trajectory approximation $\mathbf{r}_1(t) = \mathbf{r}_1 + \mathbf{v}_1 t$ that is valid at leading order. Taking the Laplace-Fourier transforms of Eqs. (148) and (149), we obtain

$$\delta\hat{f}(\mathbf{k},\mathbf{v},\omega) = \frac{\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}-\omega}\delta\hat{\Phi}(\mathbf{k},\omega), \quad (150)$$

$$\delta\hat{\Phi}(\mathbf{k},\omega) = (2\pi)^d \hat{u}(k)\int \delta\hat{f}(\mathbf{k},\mathbf{v},\omega)\,d\mathbf{v} + m\,\hat{u}(k)e^{-i\mathbf{k}\cdot\mathbf{r}_1}\delta(\mathbf{k}\cdot\mathbf{v}_1-\omega). \quad (151)$$

Substituting Eq. (150) in Eq. (151), we find that

$$\delta\hat{\Phi}(\mathbf{k},\omega) = m\,\frac{\hat{u}(k)}{\epsilon(\mathbf{k},\omega)} e^{-i\mathbf{k}\cdot\mathbf{r}_1}\delta(\mathbf{k}\cdot\mathbf{v}_1-\omega), \tag{152}$$

where $\epsilon(\mathbf{k},\omega)$ is the dielectric function (II-50). Taking the inverse Laplace-Fourier transform of Eq. (152), we obtain the effective field created by particle 1 on particle 0 taking into account collective effects:

$$\Phi(1\to 0,t) = \int m\,\frac{\hat{u}(k)}{\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_1)} e^{i\mathbf{k}\cdot(\mathbf{r}(t)-\mathbf{r}_1(t))}d\mathbf{k}. \tag{153}$$

Finally, the effective force created by particle 1 on particle 0 taking into account collective effects is

$$\mathbf{F}(1\to 0,t) = -\int im\mathbf{k}\frac{\hat{u}(k)}{\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_1)} e^{i\mathbf{k}\cdot(\mathbf{r}(t)-\mathbf{r}_1(t))}d\mathbf{k}. \tag{154}$$

References

[1] Dynamics and Thermodynamics of Systems with Long Range Interactions, edited by T. Dauxois, S. Ruffo, E. Arimondo and M. Wilkens, Lect. Not. in Phys. **602**, Springer (2002).

[2] P.H. Chavanis, Physica A **361**, 55 (2006) [Paper I].

[3] P.H. Chavanis, Physica A **361**, 81 (2006) [Paper II].

[4] P.H. Chavanis, preprint [`arXiv:0705.4405`] [Paper III].

[5] W.C. Saslaw, *Gravitational Physics of Stellar and Galactic Systems* (Cambridge Univ. Press, 1985).

[6] T. Padmanabhan, Phys. Rep. **188**, 285 (1990).

[7] P.H. Chavanis, Int J. Mod. Phys. B **20**, 3113 (2006).

[8] J. Sommeria, *Two-Dimensional Turbulence* in: New trends in turbulence, edited by M. Lesieur, A. Yaglom, F. David, Les Houches Summer School **74**, 385 (2001).

[9] P. Tabeling, Phys. Rep. **362**, 1 (2002).

[10] P.H. Chavanis, *Statistical mechanics of two-dimensional vortices and stellar systems* in [1]; See also [`cond-mat/0212223`].

[11] R. Balescu, *Statistical Mechanics of Charges Particles* (Interscience, New York, 1963).

[12] E.M. Lifshitz, L.P. Pitaevskii, *Physical Kinetics* (Pergamon Press, Oxford, 1981).

[13] T. Dauxois, V. Latora, A. Rapisarda, S. Ruffo, A. Torcini, *The Hamiltonian Mean Field Model: from Dynamics to Statistical Mechanics and back* in [1]; See also [`cond-mat/0208456`].

[14] P.H. Chavanis, *Contributions à la mécanique statistique des tourbillons bidimensionnels. Analogie avec la relaxation violente des systèmes stellaires*, Ph.D. thesis, ENS Lyon (1996).

[15] P.H. Chavanis, `[arXiv:0704.3953]`.

[16] P.H. Chavanis, Eur. Phys. J. B **52**, 61 (2006).

[17] B.B. Kadomtsev, O.P. Pogutse, Phys. Rev. Lett. **25**, 1155 (1970).

[18] G. Severne and M. Luwel, Astr. Space Sci. **72**, 293 (1980).

[19] P.H. Chavanis, *Statistical mechanics of violent relaxation in stellar systems*, in: Multiscale Problems in Science and Technology edited by N. Antonic, C.J. van Duijn, W. Jager and A. Mikelic (Springer, Berlin 2002) [astro-ph/0212205].

[20] D. Lynden-Bell, MNRAS **136**, 101 (1967).

[21] P.H. Chavanis, Physica A **359**, 177 (2006).

[22] P.H. Chavanis, Physica A **365**, 102 (2006).

[23] H. Kandrup, ApJ **244**, 316 (1981).

[24] P.H. Chavanis, J. Sommeria and R. Robert, ApJ **471**, 385 (1996).

[25] P.H. Chavanis, Physica A **332**, 89 (2004).

[26] F. Hohl, J.W. Campbell, Astron. J. **73**, 611 (1968)

[27] R. Robert and J. Sommeria, Phys. Rev. Lett. **69**, 2776 (1992)

[28] J. Sommeria, C. Staquet, R. Robert, J. Fluid Mech. **233**, 661 (1991)

[29] A. Antoniazzi, D. Fanelli, J. Barré, P.H. Chavanis, T. Dauxois, S. Ruffo, Phys. Rev. E **75**, 011112 (2007).

[30] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[31] P.H. Chavanis and C. Sire, Physica A **356**, 419 (2005).

[32] B.M. Boghosian, Phys. Rev. E **53**, 4754 (1996)

[33] H. Brands, P.H. Chavanis, R. Pasmanter, J. Sommeria, Phys. Fluids **11**, 3465 (1999)

[34] V. Latora, A. Rapisarda, C. Tsallis, Physica A **305**, 129 (2002).

[35] A. Campa, A. Giansanti, G. Morelli, [arXiv:0706.3664]

[36] A. Taruya, M. Sakagami, Phys. Rev. Lett. **90**, 181101 (2003)

[37] Y.Y. Yamaguchi, J. Barré, F. Bouchet, T. Dauxois, S. Ruffo, Physica A **337**, 36 (2004).

[38] A. Pluchino, V. Latora, A. Rapisarda, Physica D **193**, 315 (2004).

[39] P.H. Chavanis, Eur. Phys. J. B **53**, 487 (2006).

[40] J. Binney, S. Tremaine, *Galactic Dynamics* (Princeton Series in Astrophysics, Princeton, 1987).

[41] P. Mineau, M.R. Feix, J.L. Rouet, Astron. Astrophys. **228**, 344 (1990)

[42] H. Morita, K. Kaneko, Phys. Rev. Lett. **96**, 050602 (2006)

[43] F. Bouchet, T. Dauxois, Phys. Rev. E **72**, 5103 (2005).

[44] P.H. Chavanis, Eur. Phys. J. B **52**, 47 (2006).

[45] S. Chandrasekhar, ApJ **99**, 47 (1944).

[46] H. Kandrup, Astro. Space. Sci. **97**, 435 (1983).

[47] J. Hubbard, Proc. Roy. Soc. (London) A **260**, 114 (1961).

[48] S. Chandrasekhar, ApJ **97**, 255 (1943).

[49] A. Lenard, Ann. Phys. (N.Y.) **10**, 390 (1960).

[50] R. Balescu, Phys. Fluids **3**, 52 (1960).

[51] F. Bouchet, Phys. Rev. E **70**, 036113 (2004).

[52] P.H. Chavanis, J. Vatteville, F. Bouchet, Eur. Phys. J. B **46**, 61 (2005).

[53] P.H. Chavanis, Physica A **377**, 469 (2007).

[54] P. Valageas, Physical Review E **74**, 016606 (2006).